Hitotsubashi ICS-FS Working Paper Series

FS-2011-E-004

Dynamic Optimal Pension Fund Portfolios when Risk Preferences are Heterogeneous among Pension Participants

Toshiki Honda

Graduate School of International Corporate Strategy,
Hitotsubashi University

First version: November 24, 2008

Current version: April 5, 2011

Dynamic Optimal Pension Fund Portfolios when Risk Preferences are Heterogeneous among Pension Participants

Toshiki Honda*

April 5, 2011

Abstract

In this paper, we consider the problem of an optimal pension fund portfolio given the heterogeneous risk preferences of pension fund participants. The relative risk aversion of a pension fund tends to be a decreasing function of the level of aggregate wealth. We find that the dynamic optimal portfolio is simply characterized as the weighted sum of the optimal portfolio for each participant. Our model helps successfully establish the microfoundation of asset liability management models. A numerical example using recent Japanese data indicates the significant total welfare losses of adopting a suboptimal portfolio strategy and an inefficient risk-sharing rule.

*Graduate School of International Corporate Strategy, Hitotsubashi University. 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo, 101-8439, JAPAN. email : thonda@ics.hit-u.ac.jp The author is indebted to Chiaki Hara, Fumio Hayashi, Shinsuke Ikeda, Hidetoshi Nakagawa, Kazuhiko Ohashi, Akihiko Takahashi, conference attendees at the spring 2009 meeting of the Japanese Economic Association and the 2009 annual meeting of the Nippon Finance Association, and seminar participants at Hitotsubashi University for helpful comments and suggestions.

Dynamic Optimal Pension Fund Portfolios when Risk Preferences are Heterogeneous among Pension Participants

April 5, 2011

Abstract

In this paper, we consider the problem of an optimal pension fund portfolio given the heterogeneous risk preferences of pension fund participants. The relative risk aversion of a pension fund tends to be a decreasing function of the level of aggregate wealth. We nd that the dynamic optimal portfolio is simply characterized as the weighted sum of the optimal portfolio for each participant. Our model helps successfully establish the microfoundation of asset liability management models. A numerical example using recent Japanese data indicates the signi cant total welfare losses of adopting a suboptimal portfolio strategy and an ine cient risk-sharing rule.

1 Introduction

A dwindling birthrate and an aging population are serious concerns in many countries. The key challenge in an aging society is to make the pension system sustainable. That is, pension fund reserves need to be sufficiently large to fill the gap between future cash inflows and outflows. For example, the Japanese public pension fund is carrying forward more than 100 trillion Japanese yen as a reserve as of March 2010. Just as importantly, pension funds typically invest their reserves in asset markets. Accordingly, asset market returns are also an essential component of a sustainable pension system.

Pension participants typically delegate portfolio management to the pension fund. Portfolio decisions made by pension funds typically start with actuarial calculation of the required return. A portfolio is then chosen such that this required expected return is attained with the smallest risk using the mean–variance approach in Markowitz [12]. Of particular note is that these portfolio strategies implicitly assume that the risk aversion parameter is constant and unrelated to the level of wealth.

It is not surprising, then, that participants in pension systems do not easily agree about the preferred portfolio strategy. Pension participants are heterogeneous, particularly in large public pension funds. Importantly, the risk attitudes of pension participants may be heterogeneously distributed, not only because of differences in their inherent risk tolerance, but also because of a diverse personal profile, including age, income, the number of children, and so on. Pension participants will also have their own views on the future behavior of asset markets and the economy.

This article discusses the optimal portfolio strategy of pension funds by focusing on the variation in risk tolerances and the different probability assessments of pension participants. Portfolio management is delegated to a pension fund. For simplicity, we assume there is a time horizon for investment and participants are distributed reserves at some terminal date. In addition, there is no intermediate payoff before the terminal date. Pension participants are then ultimately interested in the payoff at the terminal date and have the same time discount rate.

We consider the optimal portfolio problem of pension funds as a syndicate problem. We define a syndicate as a group of individual decision makers who must make a common decision under uncertainty, and who, as a result, will jointly receive a payoff to be shared among them. Wilson [21], for instance, considered a Pareto-optimal risk-sharing rule and showed that the objective function of a syndicate may not exist when the members have diverse probability assessments. Wilson [21] also showed that an objective function of a syndicate exists when the members have the same probability assessment.

The objective function of a syndicate also has some interesting properties from a portfolio management viewpoint. For example, Hara, Huang, and Kuzmics [8] showed that the relative risk aversion of a syndicate tends to be decreasing with respect to the level of wealth. A typical asset allocation strategy based on the mean–variance approach is then not appropriate, because risk aversion in that case is assumed to be irrelevant to the level of wealth.

Although decreasing relative risk aversion is plausible, optimal portfolio problems with such utility functions have not been extensively studied, particularly in the context of multiperiod portfolio strategy. Indeed, constant absolute or relative risk aversion assumptions imply that the risk attitude is irrelevant to the level of wealth, as shown in Pratt [13]. However, constant risk aversion assumptions make the dynamic optimal portfolio problem much simpler, as the value functions of the optimal portfolio strategies are often irrelevant to the wealth level.

In other words, if we do not assume constant absolute or relative risk aversion, dynamic optimal portfolio problem would be very difficult to solve. We show that by assuming a complete security market, the optimal portfolio in our model is characterized as a weighted average of the optimal portfolio of each fund member. Our investment strategy is thus as simple as the standard mean–variance approach. However, as the weight depends on each member's optimal wealth path, the portfolio strategy is not a trivial stochastic process.

Of comparable importance are the ramifications for asset liability management (ALM). Many studies have already pointed out the importance of ALM, for example, Sharpe [16], Sharpe and Tint [17], and Bodie [1]. By assuming some of the participants seriously dislike taking risks and by considering them as "creditors", the model in this paper provides a satisfactory response to the asset liability management problem. This is because we construct the objective function in our model from the preferences of participants, in contrast to conventional ALM models, which assume exogenously given objective functions. The constructed objective function is then decreasingly relatively risk averse; this assumption is often made in the ALM literature, but with little theoretical grounding.

Our model is also useful for study of the efficient risk-sharing problem. In general, when the members of a syndicate are heterogeneous, the risk-sharing rule cannot be solved analytically. In our model, the efficient risk-sharing rule and utility function of a syndicate are easily characterized through the optimal portfolio strategy of each member.

The practical implementation of our model is also simple. To resolve the heterogeneity of risk preference among participants, the entire fund should be divided into smaller funds so that each component fund undertakes a portfolio strategy reflecting the risk tolerances of each group. Our results thus support 401K and/or personal accounting systems because these systems allow each participant to choose the appropriate level of portfolio risk reflecting their own personal risk tolerance.

In the following, we formulate the problem of a public pension fund portfolio as a syndicate problem. In Sections 2.2 and 2.3, we discuss the existence of the objective function and its properties. Section 3.1 formulates a model of the security market and optimal portfolio problem, while Section 3.2 characterizes the optimal portfolio. We define the utility function of a syndicate in Section 3.3 and the relation between our model and ALM models in Section 3.4. Using the Japanese public pension system as an example, we then consider the optimal portfolio problem. We formulate the model itself in Section 4.1, and then determine the model parameters in Section 4.2 and study the welfare loss for a typical strategy in Section 4.3.

2 Objective Function of a Pension Fund with Heterogeneous Participants

2.1 Risk-sharing Rule

We consider the pension fund's portfolio selection problem as a syndicate problem. Wilson [21] refers to a group of agents as a syndicate when they make a decision under uncertainty and distribute the entire payoff among the included agents. Agents $i = 1, 2, \ldots, I$ participate in the pension fund and make an initial contribution $W^i(0)$ at an initial time $t = 0$. The pension fund then invests the aggregated contribution $\sum_{i=1}^{I} W^i(0)$ in security markets and chooses a portfolio strategy until time T. At the terminal time $T(> 0)$, the pension fund distributes the realized wealth W to the pension participants. The pension fund as a syndicate decides the optimal portfolio strategy under security market uncertainty and distributes the terminal wealth among the pension participants.

Although the pension fund chooses a dynamic portfolio strategy for $t \in ([0,T])$, we can transform the dynamic portfolio problem into a static problem using a martingale approach. We can thus write the terminal wealth W of the pension fund as a function of the portfolio strategy φ and the state price density π, that is, $W = W(\pi, \varphi)$.

The pension fund's two decision problems, portfolio selection and wealth distribution, are easy if agents are homogeneous. If agents are heterogeneous in some respect, these two problems can be seriously complicated. In the following, we focus on the heterogeneity of risk preferences among agents. We also consider how heterogeneous views about asset returns affect the objective function.

Agent i has an expected utility function:

$$E^i(u_i(w)) = \int u_i(w)\eta_i(\pi)\, d\pi,$$

where $\eta_i(\cdot)$ is agent i's subjective probability density function for the state price deflator π, and $E^i(\cdot)$ is an expectation with respect to $\eta_i(\cdot)$. The expected utility $u_i(\cdot)$ is increasing, twice continuously differentiable, and strictly concave ($u_i'' < 0$). Absolute risk aversion is defined by:

$$t_i(w) = -\frac{u_i'(w)}{u_i''(w)},$$

which may not be the same for all i because of heterogeneous risk attitudes. The subjective probability densities $\eta_i(\cdot)$ also may not be the same if agents have different views on market investment opportunities.

A risk-sharing rule is a set of functions $f = \{f_i(w,\pi) | i = 1,\dots,I\}$ such that $\sum_i f_i(w,\pi) = w$ for all (w,π). If a risk-sharing rule f depends on portfolio φ, then we write $f(\varphi) = \{f_i(w,\pi|\varphi) | i = 1,\dots,I\}$. A risk-sharing rule f is then Pareto optimal if there is no other allocation that improves at least one agent's expected utility without reducing the expected utility of other agents.

It is well known that for any Pareto-optimal sharing rule $f(\varphi)$, there are nonnegative weights $\{\lambda_i(\varphi) | i = 1,\dots,I\}$ so that $f(\varphi)$ is a solution to the following variational problem:

$$\begin{aligned} \max \quad & \sum_i \lambda_i(\varphi) \int u_i(f_i(w,\pi), \pi|\varphi)\eta_i(\pi)\, d\pi \\ \text{s.t.} \quad & \sum_i f_i(w,\pi|\varphi) = w \quad \forall (w,\pi|\varphi). \end{aligned} \tag{1}$$

The following proposition shows the necessary and sufficient condition for the Pareto optimality of the risk-sharing rule as a differential condition of the variational problem.

Proposition 1 (Wilson [21], Theorem 1.)**.** *A necessary and sufficient condition for Pareto optimality of the sharing rule $f(\varphi)$ is that there exist nonnegative weights $\lambda(\varphi) = \{\lambda_i(\varphi)\}$ and a function $y_0(w,\pi|\varphi)$ such that:*

1. $\sum_i f_i(w, \pi|\varphi) = w \quad \forall(w, \pi|\varphi)$

2. $\lambda_i(\varphi)u_i'(f_i(w, \pi|\varphi)) = y_0(w, \pi|\varphi)$ *for each* i *and* π *almost everywhere for which* $\lambda_i(\varphi)f_i(\pi) > 0$.

The Lagrangean of variational Problem (1) is given by:

$$
\begin{aligned}
&\mathcal{L}(f(\varphi), y_0(W(\pi, \varphi), \pi|\varphi)) \\
= &\int \left\{ \sum_i [\lambda_i(\varphi)u_i(f_i(w(\pi, \varphi), \pi|\varphi))\eta_i(\pi) - f_i(w(\pi, \varphi), \pi|\varphi)y_0(W(\pi, \varphi), \pi|\varphi)] + W(\pi, \varphi)y_0(W(\pi, \varphi), \pi|\varphi) \right\} d\pi,
\end{aligned}
$$

That is, $y_0(\cdot|\cdot)$ in Proposition 1 is a Lagrange multiplier of Problem (1). We write the Pareto-optimal risk-sharing rule given a weight $\lambda = \{\lambda_i(\varphi)|i = 1, \ldots, I\}$ as $f^\lambda(\varphi) = \{f_i^\lambda(w, \pi|\varphi)|i = 1, \ldots, I\}$. The portfolio φ should be chosen so that the Lagrangean $\mathcal{L}(\cdot, \cdot)$ of the variational problem is maximized under the budget constraint. We explicitly provide the portfolio problem in Section 3 with the description of the security market model. Before doing so, however, we summarize the known properties of risk-sharing rules and objective functions.

2.2 Existence of the Objective Function

To find a solution to the optimal portfolio problem, the objective function of pension fund should be well defined. Wilson [21] shows the conditions under which the objective function of a syndicate exists. For each agent i, define a function $M_i(\cdot, \cdot)$ by $M_i(w, \pi) = u_i(w)\eta_i(\pi)$. Suppose that agent i chooses portfolio φ_i, which yields terminal wealth $W_i(\pi, \varphi_i)$. $M_i(\cdot, \cdot)$ is then said to be an evaluation measure of i if the selected portfolio φ_i maximizes:

$$\int M_i(W_i(\pi, \varphi_i), \pi)\, d\pi.$$

An evaluation measure $M_\lambda(W(\pi, \varphi), \pi)$ for a syndicate is similarly defined. If there exists an evaluation measure $M_\lambda(\cdot, \cdot)$, pension participants may ask the pension fund to maximize $\int M_\lambda(W(\pi, \varphi), \pi)\, d\pi$ by choosing a portfolio strategy φ. If an evaluation measure $M_\lambda(w, \pi)$ is separable and $M_\lambda(w, \pi) = u_\lambda(w)\eta(\pi)$, we refer to $u_\lambda(\cdot)$ and $\eta(\cdot)$ as the surrogate utility and probability functions, respectively. An evaluation measure $M_\lambda(w, \pi)$ exists if the weights $\{\lambda_i(\varphi)\}$ do not depend on portfolio φ. (See Wilson [21], Theorem 2.) It is thus meaningless to discuss optimal portfolio strategies unless the utility weights $\{\lambda(\varphi)\}$ are determined beforehand.

Wilson's [21] Theorem 3 points out that the necessary and sufficient condition for the existence of a syndicate's surrogate functions is that Savage's sure-thing principle (independence axiom, Savage [15] p. 31, Kreps [11] p. 103) is satisfied. Importantly, pension funds cannot locate the optimal portfolio if the "probability" distribution does not satisfy the axioms of probability. Further, the realized pension fund return cannot be evaluated by the fund participants.

When agents have different views about investment opportunities, the surrogate functions of a syndicate do not always exist. Wilson's [21] Theorems 6, 7, and 9, for instance, show that the surrogate function exists when the sharing rule is linear, that is, the first derivative of the sharing rule is independent of the wealth level w. This occurs if all agents have constant absolute risk aversion utility functions. On the other hand, surrogate functions exist when all agents have the same view on investment opportunities and $\eta_i(\cdot)$ is identical for all i.

Unfortunately, both assumptions—homogeneous forecasts and linear sharing rules—are strong assumptions that may not be satisfied in reality. However, we do assume forecasts are homogeneous in the following manner. Namely, if it is not possible for all agents to agree on a particular probability distribution $\eta(\cdot)$, it may be innocuous to assume that agents agree to accept expert opinion on investment opportunities given by investment professionals.

2.3 Properties of the Surrogate Utility Function

We suppose that all agents have the same probability distribution $\eta(\cdot)$. It follows from Wilson's [21] Theorems 6 and 7 that there exists a surrogate utility function $u_\lambda(\cdot)$ of a syndicate. Given weights $\lambda = \{\lambda_i | i = 1, \ldots, I\}$, the surrogate utility function $u_\lambda(\cdot)$ is defined by:

$$u_\lambda(w) = \sum_i \lambda_i E[u_i(f_i^\lambda(w))],$$

where $f^\lambda(\cdot)$ is a solution to variational Problem (1) and $E[\cdot]$ is an expectation with respect to $\eta(\cdot)$.

For each agent i, absolute risk aversion (ARA) $a_i(w)$, absolute risk tolerance (ART) $t_i(w)$, relative risk aversion (RRA) $b_i(w)$, and relative risk tolerance (RRT) $s_i(w)$ are defined as follows:

$$\begin{aligned} a_i(w) &= -\frac{u_i''(w)}{u_i'(w)}, \qquad t_i(w) = \frac{1}{a_i(w)}, \\ b_i(w) &= -\frac{u_i''(w)w}{u_i'(w)}, \qquad s_i(w) = \frac{1}{b_i(w)}. \end{aligned}$$

The first derivative $t_i'(w)$ of the ART is known as cautiousness. For a surrogate utility function $u_\lambda(\cdot)$ of a syndicate, ARA $a_\lambda(\cdot)$, ART $t_\lambda(\cdot)$, RRA $b_\lambda(\cdot)$, and RRT $s_\lambda(\cdot)$ are similarly defined. The following are the basic properties of $u_\lambda(\cdot)$ and the sharing rule.

Proposition 2 (Wilson [21], Theorem 4 and Theorem 5). *1.* $t_\lambda(w) = \sum_i t_i(f_i^\lambda(w))$

2.

$$\frac{\partial f_i^\lambda(w,\pi)}{\partial w} = \frac{t_i(f_i^\lambda(w,\pi))}{t_\lambda(w,\pi)}$$

ART $t_i(x)$ is half of the variance accepted by receiving a unit risk premium (see Pratt [13]). The first property in Proposition 2 shows that the syndicate's ART, that is, the variance accepted by the syndicate, is the sum of the ART of each agent i. The second property shows that a marginal increase in total wealth is distributed to each agent i by the ratio of accepted variance by agent i and the syndicate.

The following propositions describe the property of ART $t_\lambda(\cdot)$ for the syndicate.

Proposition 3 (Hara, Huang and Kuzmics [8], Theorem 4)**.**

$$t''_\lambda(w) = \sum_i \left(\frac{\partial f_i^\lambda(w,\pi)}{\partial w}\right)^2 t''_i(f_i^\lambda(w,\pi)) + \frac{1}{w}\sum_i \left(\frac{\partial f_i^\lambda(w,\pi)}{\partial w}\right)(t'_i(f_i^\lambda(w)) - t'_\lambda(w))^2$$

It follows from Proposition 3 that if the ART of each i is convex ($t''_i(w) \geq 0$ for all i), the syndicate's ART is also convex ($t''_\lambda(w) \geq 0$). We rephrase these properties using the ARA in the next proposition.

Proposition 4 (Hara, Huang and Kuzmics [8], Corollary 7)**.** *If RRA $b_i(\cdot)$ of i is a nonincreasing function of every i, then $b_\lambda(\cdot)$ is nonincreasing. If, moreover, $\{b_i(\cdot)\}$ are not completely equal at any aggregate consumption level, then $b_\lambda(\cdot)$ is strictly decreasing.*

Therefore, if agents have different risk attitudes, the relative risk aversion of the syndicate's utility is decreasing (DRRA). The intuition behind this proposition is clear. By the second part of Proposition 2, a marginal increase in total wealth is distributed more to less risk-averse agents. The shares of these less risk-averse agents then increase, which implies that the risk tolerance $t_\lambda(w)$ of the syndicate is higher given the first part of Proposition 2. Thus, the RRA of the syndicate is decreasing with respect to total wealth.

The fact that the RRA of the syndicate tends to be decreasing raises a question about a generally accepted portfolio strategy that allocates total wealth to some asset classes at a constant fraction. This strategy is supported if not only investment opportunities are constant but also the ARA or RRA is constant. However, we show here that the CARA and CRRA utilities may not be appropriate as an objective function for a pension fund with heterogeneous participants.

3 Optimal Portfolio Strategy

Without assuming CARA or CRRA utility functions, optimal portfolio problems are difficult to solve in a multiperiod or continuous time setting. For DRRA utility functions, which are more plausible for pension funds with heterogeneous participants, the attitude toward risk depends on the level of wealth. However, the value function of the portfolio problem may not be separable with respect to wealth and investment opportunity. Thus, finding an analytical solution to the optimal portfolio problem may be more difficult.

In our model, the optimal portfolio for the DRRA utility function can be characterized through the optimal portfolio strategy for each agent. By assuming each participant has a simple utility function, such as constant relative risk aversion, we can easily characterize the decreasing relative risk aversion utility function of the pension fund using the optimal portfolio for CRRA utility. In this section, we first provide a security market model, find the optimal portfolio strategies, and characterize these using value functions.

3.1 Security Market and Optimal Portfolio Problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $B := (B_1, \dots, B_d)^\top$ be d-dimensional standard Brownian motion, and $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ be an augmented Brownian filtration. A vector $X(t) = (X_1(t), \dots, X_n(t))$ of state variables describes investment opportunities of market and satisfies the following stochastic differential equation:

$$X(t) = x + \int_0^t \mu^X(s, X(s))\, ds + \int_0^t \sigma^X(s, X(s))\, dB(s),$$

where $x = (x_1, \dots, x_n) \in R^n$, $\mu^X : [0,T] \times R^n \to R^n$, $\sigma^X : [0,T] \times R^n \to R^n \otimes R^d$, and:

$$\mu^X(t,x) = \begin{pmatrix} \mu_1^X(t,x) \\ \vdots \\ \mu_n^X(t,x) \end{pmatrix}, \qquad \sigma^X(t,x) = \begin{pmatrix} \sigma_{1,1}^X(t,x) & \cdots & \sigma_{1,d}^X(t,x) \\ \vdots & & \vdots \\ \sigma_{n,1}^X(t,x) & \cdots & \sigma_{n,d}^X(t,x) \end{pmatrix}.$$

A function $r : [0,T] \times R^n \to R$ defines the riskless rate by $r_t = r(t, X(t))$, and the money market account S_0 is given by:

$$S_0(t) = S_0(0) \exp\left(\int_0^t r(s, X(s))\, ds \right).$$

There are d risky securities $S_j(t)$, $j = 1, \dots, d$, which satisfy:

$$S_j(t) = S_j(0) + \int_0^t \mu_j(s, X(s)) S_i(s)\, ds + \sum_{k=1}^d \int_0^t \sigma_{j,k}(s, X(s)) S_j(s)\, dB_k(s), \quad j = 1, \dots, d.$$

Furthermore we write $S(t) = (S_1(t), \dots, S_d(t))$,

$$\mu(t,x) = \begin{pmatrix} \mu_1(t,x) \\ \vdots \\ \mu_d(t,x) \end{pmatrix}, \quad \text{and} \quad \sigma(t,x) = \begin{pmatrix} \sigma_{1,1}(t,x) & \cdots & \sigma_{1,d}(t,x) \\ \vdots & & \vdots \\ \sigma_{d,1}(t,x) & \cdots & \sigma_{d,d}(t,x) \end{pmatrix}.$$

The volatility matrix $\sigma(t,x)$ is assumed to be invertible for any $t \in [0,T]$ and any $x \in R^n$. Thus, the market is complete and a function $\theta : [0,T] \times R^n \to R^d$ is defined by:

$$\theta(t,x) = (\theta_1(t,x), \ldots, \theta_d(t,x)) = \sigma(t,x)^{-1}(\mu(t,x) - r(t,x)).$$

The state price deflator $\pi(t)$ is defined by:

$$\pi(t) = \pi(0) \exp\left[-\int_0^t r(s,X(s))\, ds - \int_0^t \sum_{j=1}^d \theta_j(s,X(s))\, dB_j(s) - \frac{1}{2}\int_0^t \sum_{j=1}^d \theta_j(s,X(s))^2\, ds \right].$$

The utility function $u_\lambda(\cdot)$ is defined by

$$u_\lambda(w) = \sum_i \lambda_i E[u_i(f_i^\lambda(w))],$$

where $f^\lambda(\cdot)$ is a solution to variational Problem (1) given a weight vector $\lambda = \{\lambda_i | i = 1, \ldots, I\}$. The pension fund maximizes the utility function $u_\lambda : R_+ \to R$ subject to the budget constraint. Given $\mathcal{F}_t$-measurable portfolio strategy $\varphi(t) \equiv (\varphi_1(t), \ldots, \varphi_d(t))^\top$, the wealth process W satisfies:

$$dW(t) = W(t)\left[\left(r(t) + \varphi(t)^\top(\mu(t) - r(t))\right)\right] dt + W(t)\varphi(t)^\top \sigma(t)\, dB_t, \qquad W(0) = W_0. \tag{2}$$

Let Θ be a set of admissible portfolio strategies, that is, strategies that make $W(t) \geq 0$ for any $t \in [0,T]$. The optimal portfolio problem is then defined by:

$$\begin{aligned} \max_{\varphi \in \Theta} \quad & E\left[u_\lambda(W(T))\right] \\ \text{s.t.} \quad & (2) \end{aligned} \tag{3}$$

Given the market is complete, we can apply the martingale approach by Cox and Huang [5] and Karatzas and Shreve [9]:

$$\begin{aligned} \max_{W(T)} \quad & E\left[u_\lambda(W(T))\right] \\ \text{s.t.} \quad & E\left[\frac{\pi(T)}{\pi(0)} W(T)\right] \leq W_0. \end{aligned} \tag{4}$$

Let $W^*(T)$ be a solution to Problem (4).

To characterize a solution $W^*(T)$ to Problem (4), we consider the problem:

$$\begin{aligned} \max_{\{W_i(T)|i=1,\ldots,I\}} \quad & E\left[\sum_i \lambda_i u_i(W_i(T))\right] \\ \text{s.t.} \quad & E\left[\frac{\pi(T)}{\pi(0)}\left(\sum_i W_i(T)\right)\right] \leq W_0. \end{aligned} \tag{5}$$

Problem (5) maximizes the weighted average of utility functions $u_i(\cdot)$ by choosing wealth $\{W_i(T)|i = 1, \ldots, I\}$ for each i under the budget constraint. Let $\{W_i^*(T)|i = 1, \ldots, I\}$ be a solution to Problem

(5); then, the solution $W^*(T)$ to Problem (4) is obtained from $\{W_i^*(T)\}$. Furthermore, we can find the solution $\{W_i^*|i = 1, \dots, I\}$ to Problem (5) by redistributing the initial total wealth to agents and making agents find their own optimal portfolio.

Proposition 5. *The solution $W^*(T)$ to Problem (4) is given by the solution $\{W_i^*(T)|i = 1, \dots, I\}$ to Problem (5) by setting $W^*(T) = \sum_i W_i^*(T)$. Define an initial wealth distribution $\{\tilde{W}_i^0|i = 1, \dots, I\}$ to agents by:*

$$\tilde{W}_i^0 = E\left[\frac{\pi(T)}{\pi(0)} W_i^*(T)\right], \qquad i = 1, 2, \dots, I. \tag{6}$$

The solution $\{W_i^*(T)\}$ to Problem (5) is then given as a solution to the following problem for each agent $i = 1, \dots, I$:

$$\begin{aligned} \max_{W_i(T)} \quad & E\left[u_i(W_i(T))\right] \\ \text{s.t.} \quad & E\left[\frac{\pi(T)}{\pi(0)} W_i(T)\right] \leq \tilde{W}_i^0. \end{aligned} \tag{7}$$

Proof: According to the saddle point theorem, there is a Lagrange multiplier $y \geq 0$ such that $\{W_i^*(T)|i = 1, \dots, I\}$ solves Problem (5):

$$\sup_{\{W_i(T)|i=1,\dots,I\}} E\left[\sum_i \lambda_i u_i(W_i(T))\right] + y\left(W_0 - E\left[\frac{\pi(T)}{\pi(0)}\left(\sum_i W_i(T)\right)\right]\right). \tag{8}$$

Given that $\{f_i^\lambda(\sum_i W_i^*)|i = 1, \dots, I\}$ is a sharing rule and satisfies the budget constraint of (5):

$$\begin{aligned} & E\left[\sum_i \lambda_i u_i(W_i^*(T))\right] + y\left(W_0 - E\left[\frac{\pi(T)}{\pi(0)}\left(\sum_i W_i^*(T)\right)\right]\right) \\ \geq \; & E\left[\sum_i \lambda_i u_i\left(f_i^\lambda\left(\sum_i W_i^*(T)\right)\right)\right] + y\left(W_0 - E\left[\frac{\pi(T)}{\pi(0)}\left(\sum_i f_i^\lambda\left(\sum_i W_i^*(T)\right)\right)\right]\right). \end{aligned}$$

By the fact that $\sum_i W_i^*(T) = \sum_i f_i^\lambda(\sum_i W_i^*(T))$, we have:

$$E\left[\frac{\pi(T)}{\pi(0)}\left(\sum_i W_i^*(T)\right)\right] = E\left[\frac{\pi(T)}{\pi(0)}\left(\sum_i f_i^\lambda\left(\sum_i W_i^*(T)\right)\right)\right]$$

and thus:

$$E\left[\sum_i \lambda_i u_i(W_i^*(T))\right] \geq E\left[\sum_i \lambda_i u_i\left(f_i^\lambda\left(\sum_i W_i^*(T)\right)\right)\right]. \tag{9}$$

On the other hand, as the risk-sharing rule $\{f_i^\lambda\}$ is a solution to variational Problem (1), we also have:

$$E\left[\sum_i \lambda_i u_i(W_i^*(T))\right] \leq E\left[\sum_i \lambda_i u_i\left(f_i^\lambda\left(\sum_i W_i^*(T)\right)\right)\right].$$

Therefore, we can verify that the solution $W^*(T)$ to Problem (4) is given by the solution $\{W_i^*(T)\}$ to Problem (5) by setting $W^*(T) = \sum_i W_i^*(T)$.

Let $\tilde{W}_i(T)$ be the solution to Problem (7). Suppose that $E[u_i(\tilde{W}_i(T))] > E[u_i(W_i^*(T))]$ for agent i. Then:

$$\sum_i \lambda_i E[u_i(\tilde{W}_i(T))] > \sum_i \lambda_i E[u_i(W_i^*(T))].$$

Furthermore, $\{\tilde{W}_i(T)\}$ satisfies the budget constraint of Problem (5), which contradicts the fact that $\{W_i^*(T)\}$ is a solution to Problem (5). $\square$

We are interested in the optimal portfolio for Problem (3), that is, the portfolio that attains a solution $W^*(T)$ to Problem (4). It follows from Proposition 5 that the portfolio that attains $W^*(T)$ is obtained by finding the portfolio that attains a solution to Problem (7). The optimal portfolio is then defined by the weighted average of the portfolio of each agent, where the weight is given by the associated wealth process to Problem (7). This result is summarized in the following proposition.

Proposition 6. *Let $\tilde{W}_i(T)$ be the optimal wealth for Problem* (7) *and let $\tilde{\varphi}_i$ be the portfolio process that attains $\tilde{W}_i(T)$. Define the wealth process $\tilde{W}_i(t)$, $t \in [0, T]$, from the portfolio strategy $\tilde{\varphi}_i$. The optimal portfolio φ^* of an investor with utility function $u_\lambda(\cdot)$ is then a weighted average of the optimal portfolio $\tilde{\varphi}_i$ of an agent i given an initial asset $\tilde{W}_i(0)$ in* (6)*, where the weight is given by the optimal wealth process $\tilde{W}^i$:*

$$\varphi^*(t) = \sum_{i=1}^{I} \left(\frac{\tilde{W}_i(t)}{\sum_{j=1}^{I} \tilde{W}_j(t)} \tilde{\varphi}_i(t) \right). \tag{10}$$

We can thus find the optimal portfolio for a DRRA utility function as a weighted average of the optimal portfolio of each agent. However, this result is only useful when each agent's optimal portfolio strategy is known. For practical purposes, it may be more reasonable to assume that each agent i has a CRRA utility function so that we can find the optimal portfolio for each agent. Even if each has a CRRA utility function, the whole group tends to be DRRA, as we have already observed.

The optimal portfolio in Equation (10) is a weighted sum of each agent's optimal portfolio. However, the portfolio strategy φ^* is not as simple as it appears, because the wealth path $\tilde{W}_i$ of each agent is stochastic and the weight $\tilde{W}_i / \sum_{j=1}^{I} \tilde{W}_j$ is not a straightforward stochastic process.

3.2 Value Function and the Optimal Portfolio

Based on the results shown in the previous sections, we now study the optimal portfolio and the associated value function in more detail. We first write down the optimal portfolio problem corresponding to Problem (7), that is, the optimal portfolio problem of each i given initial wealth $\tilde{W}_i^0$:

$$\begin{aligned} \max_{\varphi^i \in \Theta} \quad & E\left[u_i(\tilde{W}_i(T))\right] \\ \text{s.t.} \quad & d\tilde{W}_i(t) = \tilde{W}_i(t)\left[\left(r(t) + \varphi^i(t)^\top(\mu(t) - r(t))\right)\right] dt + \tilde{W}_i(t)\varphi^i(t)^\top\sigma(t)\, dB_t, \quad \tilde{W}^i(0) = \tilde{W}_i^0. \end{aligned} \tag{11}$$

The Hamilton–Jacobi–Bellman (HJB) equation for this problem is given by:

$$\sup_{\varphi^i} \quad \mathcal{D}V^i(w, X, t) = 0$$

with the boundary condition $V^i(w, X, T) = u_i(w)$, where:

$$\begin{aligned} \mathcal{D}V^i(w, X, t) \quad = \quad & V_w^i\left(w\left[\left(r(t) + \varphi^i(t)^\top(\mu(t) - r(t))\right)\right]\right) + V_X^i\mu^X + V_t^i \\ & + \frac{w^2}{2}\varphi^{i\top}\sigma\sigma^\top\varphi^i V_{ww}^i + \frac{1}{2}\mathrm{tr}[\sigma^X\sigma^{X\top}V_{XX}^i] + w\varphi^i\sigma\sigma^{X\top}V_{Xw}^i. \end{aligned}$$

Assuming that a solution to the HJB equation exists, it follows from the first-order condition that:

$$\varphi^i = -\frac{V_w^i}{wV_{ww}^i}(\sigma\sigma^\top)^{-1}(\mu - r) - \frac{V_{Xw}^i}{wV_{ww}^i}(\sigma\sigma^\top)^{-1}\sigma\sigma^{X\top}. \tag{12}$$

The first term is the instantaneous mean–variance portfolio and the second term is the hedging portfolio.

It follows from (10) that the optimal portfolio φ^λ of the pension fund is defined using the generated wealth processes $\tilde{W}_i(t)$:

$$\varphi^\lambda = \sum_{i=1}^I \left(\frac{\tilde{W}_i(t)}{\sum_{j=1}^I \tilde{W}_j(t)}\right)\varphi^i = \left(\frac{1}{\sum_{j=1}^I \tilde{W}_j}\right)\left[\left(-\sum_{i=1}^I \frac{V_w^i}{V_{ww}^i}\right)(\sigma\sigma^\top)^{-1}(\mu - r) + \left(-\sum_{i=1}^I \frac{V_{Xw}^i}{V_{ww}^i}\right)(\sigma\sigma^\top)^{-1}\sigma\sigma^{X\top}\right].$$

The optimal portfolio of the whole group also consists of two parts: the mean–variance portfolio and the hedging portfolio. The mean–variance portfolio is determined as if we take

$$-\frac{1}{\sum_i \tilde{W}_i}\left(\sum_i \frac{V_w^i}{V_{ww}^i}\right)$$

as the RRT of the pension fund. Here, $-V_w^i/V_{ww}^i$ can be considered as the ART with respect to the value function of each agent. Further, the ART of the pension fund is defined as a sum of the ART for each i participant. This property is similar to the property described in Proposition 2.

The hedging portfolio is more difficult to characterize. The total amount of the hedging portfolio is determined by the sum of:

$$-\frac{V_{Xw}^i}{V_{ww}^i} = -\frac{V_w^i}{V_{ww}^i}\left(\frac{\partial}{\partial x}\ln V_w^i\right),$$

that is, the product of the ART and the marginal value of the growth rate. The hedging portfolio of each participant i is only known explicitly for some special cases, such as Kim and Omberg [10]. The hedging portfolio of the whole group is a weighted sum of each agent's hedging portfolio. Given the weight determined through the wealth path $\tilde{W}_i$, the hedging portfolio of the pension fund is a stochastic combination of each agent's hedging portfolio.

3.3 Constant but Heterogeneous Relative Risk Aversion Case

In general, it is not easy to find an analytical solution to variational Problem 1. For example, even if all agents have CRRA utility functions, but different risk aversion parameters, the surrogate utility function is not CRRA. The risk-sharing rule could also be found numerically, although numerical computation is difficult as the number of agents increases.

In our model, the efficient risk-sharing rule can be characterized through the optimal portfolio strategies of agents. Suppose for simplicity in this section that the investment opportunity set is constant and the utility functions of all agents are power utility functions:

$$u_i(w) = \frac{1}{1 - 1/\kappa_i} w^{1-1/\kappa_i}.$$

The riskless rate is constant, and there is one risky asset whose price S satisfies a geometric Brownian motion:

$$dS_t = \mu S_t \, dt + \sigma S_t dB_t.$$

Suppose that the initial wealth $W(0)$ is distributed to each agent i with an amount $\tilde{W}_i(0)$. As is well known, the optimal portfolio strategy in this case is given by:

$$\tilde{\varphi}_i = \kappa_i \frac{\mu - r_f}{\sigma^2}.$$

The wealth process generated by $\tilde{\varphi}_i$ is:

$$\tilde{W}_i(t) = \tilde{W}_i(0) \exp\left[\int_0^t (\tilde{\varphi}_i(\mu - r_f) + r_f)\, dt + \int_0^t \tilde{\varphi}_i \sigma \, dB_t - \frac{1}{2}\int_0^t (\tilde{\varphi}_i \sigma)^2 \, dt\right],$$

and the total wealth process is given by $\tilde{W}(t) = \sum_i \tilde{W}_i(t)$. The efficient risk-sharing rule $f^\lambda(\cdot)$ and the surrogate utility function $u_\lambda(\cdot)$ are given by:

$$u_\lambda\left(\sum_i \tilde{W}_i(T)\right) = \sum_i \lambda_i u_i(\tilde{W}_i(T)).$$

In this case, the surrogate utility function $u_\lambda(\cdot)$ of the pension fund is not the CRRA, but its portfolio is the weighted sum of optimal portfolios chosen by investors with CRRA utility functions. The RRT of the pension fund is then the weighted average of each participant's RRT.

Lemma 1. *Suppose that the investment opportunity is constant and that all agents have CRRA utility functions with RRT parameter κ_i. The optimal portfolio of the pension fund at $t \in [0,T)$ is then the same as the optimal portfolio of a CRRA agent with RRT parameter*

$$\frac{\tilde{W}_i(t)}{\sum_{j=1}^{I} \tilde{W}_j(t)} \kappa_i. \tag{13}$$

Proof. The optimal portfolio is given by:

$$\varphi_t^* = \sum_{i=1}^{I} \frac{\tilde{W}_i(t)}{\sum_{j=1}^{I} \tilde{W}_j(t)} \tilde{\varphi}_i = \sum_{i=1}^{I} \left[\frac{\tilde{W}_i(t)}{\sum_{j=1}^{I} \tilde{W}_j(t)} \kappa_i \right] \left(\frac{\mu - r_f}{\sigma^2} \right),$$

which is the same as the optimal portfolio for CRRA utility with RRT parameter (13). □

Lemma 1 does not say that $u_\lambda(\cdot)$ is the CRRA. We take the optimal portfolio φ^* as an instantaneous mean–variance portfolio and compute the implied RRT, which is a weighted average of each agent's RRT κ_i with a weight determined by wealth $\tilde{W}_i$. The implied RRT depends on the relative size of wealth $\tilde{W}_i$, and the implied RRT of $u_\lambda(\cdot)$ is closer to RRT κ_i of agent i, who has the largest wealth among the pension participants. Security market performance before time $t \in [0,T]$ is thus relevant to portfolio selection at time t. As a less risk-averse agent's wealth grows more (less) during good (bad) market conditions, the optimal portfolio tends to be more (less) aggressive after the security market has been good (bad).

3.4 Asset Liability Management

Pension funds often determine their strategic asset allocations, which are typically determined from a long-term point of view and are kept constant unless investment opportunities have clearly changed. Because strategic asset allocation is simple and transparent, it is widely used and considered as one of the most important investment decisions.

However, in the process of determining strategic asset allocation, liability is not explicitly taken into account. Many studies point out the importance of asset liability management. See, for example, Sharpe and Tint [17], Boulier, Trussant, and Florens [3], Cairns, Blake, and Dowd [4], Sundaresan and Zapatero [18], Rudolf and Ziemba [14], Haberman and Vigna [7], Binsbergen and Brandt [19], and Detemple and Rindisbacher [6].

The results in the earlier sections suggest the importance of considering not only the existence of liability but also its structure. In particular, if the pension's liabilities comprise heterogeneous participants, neither the CARA nor the CRRA utility function is appropriate. The objective function and/or constraints are exogenously given in ALM models, although these are practically and intuitively chosen. The optimization problems tend to become complicated as we depart from the simple and standard settings of the portfolio selection problem. To make the optimization problems tractable, it is often assumed in ALM models that the objective functions are homothetic, such as a quadratic cost function. These assumptions often yield rather counterintuitive solutions. For example, a solution portfolio may be irrelevant to the level of assets or the funding ratio.

The marginal contribution of this paper to the ALM literature is that our model produces a micro-foundation for ALM models. Assuming that, for example, one of the participants is very risk averse, the allocations to this agent can be considered as a payment to a creditor. The DRRA utility function is then a natural choice because investors with debt payments attempt to lessen their risk exposure when total wealth is not sufficient to fulfill the debt payment. Such an objective function is not exogenously assumed, but derived as a solution to the risk-sharing rule in our model.

Our model points out two potential sources of welfare losses that could be incurred by ignoring liability structures. The first source of loss is the fact that the realized total wealth is distributed without taking heterogeneous risk attitudes into account. The second source is the fact that the asset portfolio strategy is determined by ignoring risk- sharing rules. In the following section, we consider a practical example of the optimal portfolio and welfare loss.

4 Example — The Asset Allocation Decision of the Japanese Public Pension Fund

In this section, we consider a simple asset allocation problem for the Japanese public pension fund. The fund reserves account for more than 100 trillion Japanese yen as of March 2010. The asset allocation is currently relatively conservative with about 70 percent in domestic bonds and the remainder almost equally divided across domestic stock, foreign bonds, and foreign stock. The asset allocation is determined using a mean–variance approach, the input parameters for which are estimated using long-run historical data.

Although there has long been debate about the risk caused by security investment, what has not been discussed are risk-sharing rules for heterogeneous participants. In other words, portfolio strategy has been discussed without considering how the gain (loss) is distributed (shared) among participants. We have shown in previous sections of this study that risk-sharing rules and portfolio strategies are inseparably related.

In the following, we consider a simple example of the asset allocation problem, where the riskless rate is stochastic. Uncertainty about interest rates is particularly important for the Japanese pension system, because public pension funds mostly invest in domestic bonds. Moreover, Japan has experienced low interest rates and a deflationary economy for more than a decade.

For simplicity, we consider "young" and "old" generations. We assume these two generations have power utility functions with different relative risk aversion. Planed benefits from Japanese public pension fund to older generations are not sensitive to its performance of asset portfolio. Such benefit schedule is consistent with the assumption that older generations are strongly risk averse. Optimal portfolio strategies and risk-sharing rules for the two agents with different risk attitudes are calculated. We show that the welfare losses by adopting inefficient risk-sharing rules and suboptimal portfolio strategies using constant asset allocation can be large.

4.1 Security Market Model

Let $B = (B^1, B^2)$ be an independent two-dimensional Brownian motion. The riskless rate process is given by the Vasicek model [20] as follows:

$$dr_t = \phi(\bar{r} - r_t)\, dt + \Sigma_r\, dB_t, \tag{14}$$

where ϕ and $\bar{r}$ are constant and Σ_r is a constant vector. Without loss of generality, let $\Sigma_r = (\sigma_r, 0)$. There are two risky assets: stock and a zero-coupon bond. The stock price S satisfies:

$$dS_t = \mu_S S_t\, dt + S_t \Sigma_S\, dB_t, \tag{15}$$

where $\Sigma_S = (\rho\sigma_s, \sqrt{1-\rho^2}\sigma_s)$ is a constant vector. The stock volatility is determined by σ_s and the correlation between stock returns and the riskless rate is determined by ρ. Let $P_t^{\overline{T}}$ be the price at t of a zero-coupon bond maturing at $\overline{T}$, and write its price process as:

$$dP_t^{\overline{T}} = \mu_Z P_t^{\overline{T}}\, dt + P_t^{\overline{T}} \Sigma_Z\, dB_t,$$

where μ_Z and Σ_Z are defined below in equation (18).

We assume that the market is complete and that the market price of risk $\xi = (\xi_1, \xi_2)$ is constant. The no-arbitrage condition implies ξ is given by:

$$\begin{pmatrix} \mu_Z - r_t \\ \mu_S - r_t \end{pmatrix} = \begin{pmatrix} \Sigma_Z \\ \Sigma_S \end{pmatrix} \xi \equiv \Sigma\xi.$$

Under these assumptions, $B^* = B + \int \xi dt$ is a Brownian motion under the equivalent martingale measure. The process for the riskless rate is given by:

$$dr_t = \phi(\bar{r} - r_t)\, dt + \Sigma_r\, dB_t = \phi\left(\left(\bar{r} - \frac{\sigma_r}{\phi}\xi_1\right) - r_t\right)\, dt + \Sigma_r\, dB_t^*.$$

Let $\bar{r}^{RN} \equiv \bar{r} - (\sigma_r/\phi)\xi_1$ be the mean-reverting level under the equivalent martingale measure; then, we can apply the well-known results from Vasicek [20]. As a derivative security on the riskless rate, the zero-coupon bond price $P_t^{\overline{T}}$ is given by:

$$P_t^{\overline{T}} = e^{-a(\overline{T}-t)-b(\overline{T}-t)r},$$

where $a : [0, \overline{T}] \to \mathbb{R}$ and $b : [0, \overline{T}] \to \mathbb{R}$ are defined by:

$$a(\tau) = \left(\overline{r}^{RN} - \frac{\sigma_r^2}{2\phi^2} \right) (\tau - b(\tau)) + \frac{\sigma_r^2}{4\phi} b(\tau)^2, \tag{16}$$

$$b(\tau) = \frac{1}{\phi}(1 - e^{-\phi\tau}). \tag{17}$$

It follows from Ito's lemma that the zero-coupon bond price satisfies:

$$\begin{aligned} \frac{dP_t^{\overline{T}}}{P_t^{\overline{T}}} &= \left(-b(\overline{T} - t)\phi(\overline{r} - r_t) + \frac{1}{2}(b(\overline{T} - t))^2 \sigma_r^2 + a'(\overline{T} - t) + b'(\overline{T} - t) r_t \right) dt - b(\overline{T} - t)\Sigma_r \, dB_t \\ &= (r_t - b(\overline{T} - t)\sigma_r \xi_1) \, dt - b(\overline{T} - t)\Sigma_r \, dB_t \equiv \mu_Z \, dt + \Sigma_Z \, dB_t. \end{aligned} \tag{18}$$

The stock price S satisfies:

$$dS_t = (r_t + \Sigma_S \xi) S_t \, dt + S_t \Sigma_S \, dB_t.$$

We now consider the asset allocation problem of investing in stock, the zero-coupon bond, and a riskless asset. Consider an agent who has a power utility function:

$$u(w) = \frac{w^{1-1/\kappa}}{1 - 1/\kappa}. \tag{19}$$

Optimal portfolio and risk sharing rule of pension funds are defined through optimal portfolio of this type of agent as explained in the previous sections. Let $\varphi = (\varphi_B, \varphi_S)^\top$ be the fraction invested in the bond and stock. The wealth process w^φ given portfolio φ is:

$$dw_t = [w_t \varphi_t^\top (\mu_t - r_t) + r_t w_t] \, dt + w_t^\varphi \varphi_t^\top \Sigma \, dB_t,$$

where $\mu_t \equiv (r_t - b(\overline{T} - t)\sigma_r \xi_1, r_t + \Sigma_S \xi)^\top$.

Given the initial conditions w_0 and r_0, the optimal portfolio problem for agent i is:

$$V(w_0, r_0) \equiv \sup_\varphi E\left[u_T(w^\varphi(T)) \right]. \tag{20}$$

The HJB equation for Problem (20) is defined with respect to a function $J : \mathbb{R}_+ \times \mathbb{R} \times [0, T] \to \mathbb{R}$:

$$\begin{aligned} \sup_{\varphi \in \mathbb{R}^2} \quad & J_w(w, r, t)(w\varphi^\top(\mu - r) + rw) + J_r(w, r, t)(\phi(\overline{r} - r)) + J_t(w, r, t) + \tfrac{1}{2} J_{ww}(w, r, t) w^2 \varphi^\top \Sigma \Sigma^\top \varphi \\ & + J_{wr}(w, r, t) w \varphi^\top \Sigma \Sigma_r^\top + \tfrac{1}{2} J_{rr}(w, r, t) \Sigma_r \Sigma_r^\top = 0 \end{aligned}$$

with the boundary condition

$$J(w, r, T) = u_i(w).$$

From the first-order condition, the optimal portfolio is given by:

$$\varphi_t = -\frac{1}{w J_{ww}(w, r, t)} (\Sigma \Sigma^\top)^{-1} [J_w(w, r, t)(\mu_t - r_t) + J_{wr}(w, r, t) \Sigma \Sigma_r^\top].$$

The hedging portfolio in this example is simple: to hold the zero-coupon bond by the amount

$$\frac{J^i_{wr}(w,r,t)}{wJ^i_{ww}(w,r,t)} \times \frac{1}{b(\overline{T}-t)}$$

because

$$(\Sigma\Sigma^\top)^{-1}\Sigma\Sigma_r^\top = \begin{pmatrix} -1/b(\overline{T}-t) \\ 0 \end{pmatrix}.$$

Given an agent who has a CRRA utility function, we conjecture that the value function is separable with respect to w and r:

$$J(w,r,t) = \frac{w^{1-1/\kappa}}{1-1/\kappa} f(r,t) \tag{21}$$

for some function $f : \mathbb{R} \times [0,T] \to \mathbb{R}$. We can conjecture from the result in Kim and Omberg [10] that $f(\cdot)$ has the form

$$f(r,t) = \exp[c(T-t) + d(T-t)r]$$

for some functions $c : [0,T] \to \mathbb{R}$ and $d : [0,T] \to \mathbb{R}$ with the boundary condition $c(0) = d(0) = 0$.

Proposition 7. *The optimal portfolio is:*

$$\begin{aligned} \varphi_t &= \kappa(\Sigma\Sigma^\top)^{-1}\left[(\mu_t - r_t) + \Sigma\Sigma_r^\top \frac{f_r(r_t,t)}{f(r_t,t)}\right] \\ &= \kappa\left(\Sigma\Sigma^\top\right)^{-1}(\mu_t - r_t) + (\kappa-1)b(T-t)\begin{pmatrix} -1/b(\overline{T}-t) \\ 0 \end{pmatrix} \end{aligned} \tag{22}$$

Proof. See appendix.

The second component of the optimal portfolio is to hedge against shifts in investment opportunities. In this example, investment opportunity is determined by the riskless rate, whose changes can be hedged by holding the zero-coupon bond. The amount of zero-coupon bond needed is given by totally differentiating (21) and setting $\Delta t = 0$. The amount of wealth change needed for the riskless rate change Δr to keep the same value $\overline{V}$ is:

$$\left.\frac{dw}{dr}\right|_{V(w,r,t)=\overline{V}} = -\frac{V_r}{V_w} = -\frac{w}{1-1/\kappa} d(T-t).$$

If $\phi > 0$ and the riskless rate is mean-reverting, it follows from (17) and (27) that $dw/dr < 0$ for $\kappa < 1$. Thus, an increase (decrease) in the level of the riskless rate implies an improvement (deterioration) in investment opportunities. As the bond price increases when the riskless rate falls, wealth can be transferred to the worse state by holding the zero-coupon bond.

4.2 Parameter Values

We now employ the values that are actually used in the asset allocation decision by the Japanese public pension fund. There are five asset classes: "short-term assets", "domestic bonds", "domestic stock", "foreign bonds", and "foreign stock". The asset allocation is determined by the mean–variance approach. Tables 1 and 2 detail the means, covariances, and portfolio weights.

For simplicity, domestic stock, foreign bonds, and foreign stock are combined as a single risky security with current portfolio weights in Table 1, which we refer to simply as "stock" in the following. We also take the short-term asset as the riskless asset and consider the asset allocation problem between bonds, stock, and the riskless asset. From Table 1 and Table 2, the parameter values in our three-asset example are determined as in Table 3 and Table 4. We set stock volatility $\sigma_s = 13.16\%$, stock risk premium $\Sigma_S \xi = 4.5\% - 2\% = 2.5\%$, and the correlation between bond and stock as $\rho = 0.13$. For the term structure of interest rates, we suppose that the duration of a domestic bond is $\overline{T} - t = 5$ years and that the means and variance of the riskless asset in Table 3 are the long-run means and variance of the short rate process:

$$\begin{aligned} \lim_{T\to\infty} E_t[r_T] &= \lim_{T\to\infty} \left(\overline{r} + (r_t - \overline{r})\, e^{-\phi[T-t]}\right) = \overline{r} = 0.02, \\ \lim_{T\to\infty} \mathrm{Var}_t[r_T] &= \lim_{T\to\infty} \frac{\sigma_r^2}{2\phi}\left(1 - e^{-2\phi[T-t]}\right) = \frac{\sigma_r^2}{2\phi} = (-0.0363)^2. \end{aligned} \tag{23}$$

We also take the mean and standard deviation of the bond in Table 3 as the drift and volatility parameters in Equation (18):

$$\begin{aligned} \overline{r} - b(\overline{T} - t)\sigma_r \xi_1 &= 0.03 \\ -b(\overline{T} - t)\sigma_r &= -0.0542. \end{aligned}$$

Then we can determine the market price of risk ξ, volatility σ_r and the mean-reverting speed ϕ of the riskless rate:

$$(\xi_1, \xi_2) = (-0.1845, 0.2143), \quad \sigma_r = 0.0475, \quad \phi = 0.8568.$$

We also suppose the current riskless rate is $0.02 (= \overline{r})$.

To have asset allocation weight $\varphi = (68\%, 27\%)^\top$ in Table 3, let $\kappa = 1/5.8$. The mean–variance portfolio with this value of κ is:

$$\begin{pmatrix} \text{Bond} & \text{Stock} & \text{Riskless Asset} \end{pmatrix} = \begin{pmatrix} 66.74\% & 28.30\% & 4.96\% \end{pmatrix}.$$

The dynamic optimal portfolio for time horizon $T = 5$ years is as a sum of the mean–variance and hedging portfolios:

$$\begin{aligned} \begin{pmatrix} \text{Bond} & \text{Stock} & \text{Riskless Asset} \end{pmatrix} &= \begin{pmatrix} 66.74\% + 82.76\% & 28.30\% + 0\% & 4.96\% - 82.76\% \end{pmatrix} \\ &= \begin{pmatrix} 149.50\% & 28.30\% & -77.80\% \end{pmatrix} \end{aligned}$$

Given that the market price of risk ξ in this example is constant, the mean–variance frontier and the capital market line move but keep the same shape. The mean–variance portfolio thus does not depend on the level of the riskless rate. The zero-coupon bond is held to hedge against shifts in the riskless rate. This result suggests the importance of a dynamic aspect, even if investment opportunities are essentially unchanged, as in this example.

The covariance between asset returns and the state variables is also important. In Table 2, domestic stock and bond returns are positively correlated, which means that interest rates and stock prices are negatively correlated. However, the relation is not necessarily robust and, in fact, is reversed during the most recent decade in Japan when interest rates were very low and the economy was experiencing deflation. The hedging portfolio could be very different from the value given in this example if the underlying covariance structure of asset returns and state variables is changed.

4.3 Welfare Loss Caused by Inefficient Risk Sharing and a Suboptimal Portfolio Strategy

We now compute the welfare losses caused by an inefficient risk-sharing rule and a suboptimal portfolio strategy. Motivated by the serious aging problem in Japan, we consider the preference heterogeneity between young and old generations.

Suppose for simplicity that two agents, young and old generations, participate in the pension fund and make an initial contribution at an initial time $t = 0$. The pension fund then invests the aggregated contribution in bond, stock, and the riskless asset and chooses a portfolio strategy until the terminal time $T = 5$ years. At the terminal time T, the pension fund distributes the realized wealth to young and old generations.

Using parameter values in section 4.2, the wealth distribution to young and old generations is numerically calculated as follows. We first determine relative risk tolerance of young and old generations. Then the initial aggregated wealth of the pension fund is distributed to young and old generations. We construct binomial trees for two-dimensional Brownian motion in the riskless rate process (14) and the stock price process (15) with time interval 1/2 year. We assume that markets are complete for the model described in section 4.1, and generate sample paths of state price deflator using binomial trees. The optimal wealth processes of young and old generations, which also yield the wealth distribution under the efficient risk sharing rule, are calculated using the martingale approach. The expected utility and its certainty equivalent are found by averaging sample path values. The utility weight $\lambda = \{\lambda_{young}, \lambda_{old}\}$ is determined by matching marginal utilities of initial wealth of young, old, and the pension fund.

In the following, we do not discuss how young and old generations initially contribute to the pension fund. It may be natural in many risk sharing problems to assume that the initial aggregated wealth is distributed to each agent so that the re-distributed wealth equals to its initial contribution. However in the context of public pension funds, in particular in aging society such as Japan, public pension system inevitably transfers some wealth from one generation to the other generation. Furthermore, it is not easy to conclude how much of the pension reserve belongs to a particular participant. We thus start our numerical example with the assumption that the initial aggregated wealth is already distributed to young and old generations.

In Japan, the ratio of population of retired age (over sixty five years old) and working age (from twenty to sixty four years old) was 1 to 3.6 in year 2000. It will be expected to become 1 to 1.9 in year 2025 and 1 to 1.4 in year 2050. Somehow consistent to these numbers, we consider three cases: The aggregated wealth of the pension fund is distributed to young and old generations in the ratio of 1:3 (Case A), 1:2 (Case B), and 1:1 (Case C).

Although we assume that risk attitudes are heterogeneous between young and old generations, it is difficult for us to know true values of partiticapant's risk aversion parameter. A generally accepted view is that younger generations are less risk averse than older generations, because younger generations have their future income and consider their human capital a riskless asset. Empirical findings are often inconsistent with theoretical predictions. For example, it is known that younger generations invest less money on stock market in Japan. Theories of life-cycle finance also point out that risk preferences depend on various factors: money on deposit, a debt, holding assets such as real estate and securities, age, labor income, and a state of health. See for example Bodie, Detemple, and Rindisbacher [2]. Furthermore, pension participants might not tell their true risk preference to pension funds once they understand pension benefit are determined by taking risk preference of participants into account.

In our numerical example, we determine risk aversion parameters in view of properties of Japanese public pension benefit. Old generation has no future labor income, and they consume distributed money at terminal date T instantly. On the other hand, the distributed money to young generation is not consumed but is forced to contribute to pension fund for the future consumption. Even if portfolio performance was not successful by time T, the pension fund would not reduce pension benefit to old generation. Reduction of benefit directly downgrades consumption status of old generation, which may produce political friction between young and old generations. Because of political pressure by older generation, public pension fund may not easily reduce pension benefit to them. Public pension benefit to older generation is thus similar to risk sharing rule for very risk averse agent in our model. It is thus reasonable to model old generation as highly risk averse agent.

In the following example, we assume that old generation has very large RRT and consider various risk parameters for young. Given distributed initial wealth and risk parameters, the optimal portfolio and efficient risk sharing rule are numerically found. We also calculate the wealth process generated by instantaneous mean-variance portfolio with RRT $\kappa = 1/5.8$, which we find in section 4.2 is consistent with the strategic asset allocation by Japanese public pension fund. Pension benefit to old generation is assumed to be same as the one using optimal portfolio and efficient risk sharing rule. Since old generation is modeled as a strongly risk averse agent, old generation receive almost constant benefit. Young generation is supposed to receive the residual amount of wealth at terminal T. This suboptimal portfolio strategy and inefficient risk sharing rule approximate the current Japanese public pension system.

Numerical results are shown in table 5. The RRT $\kappa = 1/5.8 = 0.17$ is consistent with the current mean–variance portfolio in Table 1. We set the RRT of old generation as $\kappa_{old} = 1/100$, whose mean-variance portfolio invests almost nothing in stock under parameter values defined in section 4.2. We consider five combinations of heterogeneous risk attitudes where the young generation is: the same risk aversion as the old generation with a almost 0% investment in the risky asset (Case 1); less risk averse than the old generation with a 10% investment in the risky asset (Case 2); less risk averse than the old generation with $\kappa_{young} = 1/5.8$ (Case 3); less risk averse than the old generation with a 40% investment in the risky asset (Case 4); even less risk averse than the old generation with a 75% investment in the risky asset (Case 5). The initial total wealth W_0 of the fund is distributed to the young and old generations in the ratio of 1:3 (Case A), 1:2 (Case B), and 1:1 (Case C).

For each case, we compute optimal wealth path and its certainty equivalent of young and old generations. Old generation receives almost constant wealth and the certainty equivalent through Case 1 to Case 5 is same. The certainty equivalent of young generation depends on the RRT, but those differences are marginal as far as the pension fund adopts optimal portfolio strategy and efficient risk sharing rule.

When the pension fund ignores heterogeneity of risk preferences and adopts inefficient risk sharing rule, the certainty equivalences of young generation are different from Case 1 to Case 5. The RRT of the pension fund is too risk seeking in Case 1 and Case 2, and too risk averse in Case 3 and 4. In Case 1, the certainty equivalence of young is small in any initial distribution of the aggregate wealth. The RRT of the pension fund in Case 1 is too risk seeking for both young and old generations, and the young generation undertakes almost all risk of the pension fund. Because of the inefficient risk sharing, the loss of the total values of certainty equivalent are about thirty trillion Japanese yen in Case 1. Even in the case of Case 3, where the RRT of the pension fund is same as the RRT of young generation, the total loss of certainty equivalent in Case A is more than 10 trillion yen. The total loss of certainty equivalent is smaller for Case 4 and Case 5, where young generation is assumed to be risk tolerant. This may be because the pension fund adopts the RRT that takes value between young and old generations.

Our numerical example points out the significance of asset liability management by pension fund. Portfolio decisions made by pension funds typically start with actuarial calculation of the required return. It is often criticized that pension fund invests too much in risky securities and that their assumptions on expected return are too optimistic. The relation between portfolio strategy and risk sharing among participants receives but scant attention.

We consider numerical examples in which the benefit to the older generation is not strongly linked to portfolio performance. Our example shows that the younger generation bears huge loss of welfare if the risk associated with investment is not shared with the older generation. Reducing the total risk of the portfolio can resolve this problem only if pension funds are sustainable in the long run with lower return from their portfolio investment. When pension funds need higher average return on their portfolio and inevitably invest their reserve in risky assets, it is important to make risk sharing rule efficient among participants.

5 Summary

In this paper we study the optimal portfolio problem for pension funds as a syndicate problem with heterogeneous risk attitudes among participants. The relative risk aversion of the objective function tends be a decreasing function of the fund's total reserves. In general, the dynamic portfolio problem is difficult to handle if risk aversion is not constant. We characterize the optimal portfolio as a weighted sum of each participant's optimal portfolio. If we suppose a simple utility function, such as CRRA utility for each participant, then the objective function and optimal portfolio strategy are easily characterized.

We also show that an agent with a decreasing RRA utility decreases level of risk when the level of wealth deteriorates. Such properties are assumed in typical ALM models by using an exogenously given objective function and/or constraint conditions. In our model, the objective function is derived from fundamentals of the model, but is not exogenously assumed.

A numerical example of the portfolio problem using the Japanese public pension fund clearly illustrates the importance of our model. As shown, a portfolio strategy that ignores pension liabilities yields a suboptimal solution. Total reserves should then be distributed among pension participants by taking their heterogeneity into account. This is important because a suboptimal portfolio strategy and an inefficient risk-sharing rule can cause nonnegligible welfare losses.

Appendix

Proof of Proposition 7 From the conjecture in (21), it follows from the first-order condition that:

$$\varphi_t^i = \kappa_i (\Sigma\Sigma^\top)^{-1} \left[(\mu_t - r_t) + \Sigma\Sigma_r^\top \frac{f_r(r_t, t)}{f(r_t, t)} \right].$$

Substituting this into the HJB equation, we then have the ordinary differential equation for $c(\cdot)$ and $d(\cdot)$:

$$
\begin{aligned}
c'(T-t) &= \frac{\kappa-1}{2}(\mu-r)^\top(\Sigma\Sigma^\top)^{-1}(\mu-r) \\
&\quad + \left((\kappa-1)(\mu-r)^\top\Sigma^{-1}\Sigma_r^\top + \phi\bar{r}\right) d(T-t) + \frac{\kappa}{2}\Sigma_r\Sigma_r^\top d(T-t)^2 \qquad (24) \\
d'(T-t) &= \frac{\kappa-1}{\kappa} - \phi\, d(T-t) \qquad (25)
\end{aligned}
$$

with the boundary condition $c(0) = d(0) = 0$. The solution to the ordinary differential equation is given by:

$$
\begin{aligned}
c(T-t) &= \frac{\kappa-1}{2}(\mu-r)^\top(\Sigma\Sigma^\top)^{-1}(\mu-r)(T-t) \\
&\quad + (1-1/\kappa)\left(\bar{r} + \frac{\kappa-1}{\phi}(\mu-r)^\top\Sigma^{-1}\Sigma_r^\top\right) b(T-t) \\
&\quad - \frac{\kappa}{2}\Sigma_r\Sigma_r^\top\frac{(1-1/\kappa)^2}{2\phi} b(T-t)^2, \qquad (26) \\
d(T-t) &= \frac{1-1/\kappa}{\phi}\left(1-e^{-\phi(T-t)}\right) = (1-1/\kappa)\, b(T-t). \qquad (27)
\end{aligned}
$$

References

[1] Zvi Bodie. The ABO, the PBO, and pension investment policy. *Financial Analysts Journal*, 46(5):27–34, Sep. - Oct. 1990.

[2] Zvi Bodie, Jerome Detemple, and Marcel Rindisbacher. Life-cycle finance and the design of pension plans. *Annual Review of Financial Economics*, 2009. Forthcoming.

[3] J-F. Boulier, E. Trussant, and D Florens. A dynamic model for pension funds management. *Proceedings of the 5th AFIR International Colloquium*, 1:361–384, 1995.

[4] A. J. G. Cairns, D. Blake, and K. Dowd. Stochastic lifestyling: Optimal dynamic asset allocation for defined-contribution pension plans. *Journal of Economic Dynamics and Control*, 30:843–877, 2006.

[5] J. Cox and C. F. Huang. Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory*, 49:33–83, 1989.

[6] Jérôme Detemple and Marcel Rindisbacher. Dynamic asset liability management with tolerance for limited shortfalls. *Insurance: Mathematics and Economics*, 43(3):281–294, 2008.

[7] S. Haberman and E. Vigna. Optimal investment strategies and risk measures in defined contribution pension schemes. *Insurance: Mathematics and Economics*, 31:35–69, 2002.

[8] C. Hara, J. Huang, and C. Kuzmics. Representative consumer's risk aversion and efficient risk-sharing rules. *Journal of Economic Theory*, 137:652–672, 2007.

[9] I. Karatzas and S. E. Shreve. *Methods of Mathematical Finance.* Springer-Verlag, New York, 1989.

[10] Tong Suk Kim and Edward Omberg. Dynamic nonmyopic portfolio behavior. *The Review of Financial Studies*, 9(1):141–161, 1996.

[11] D. M. Kreps. *A Course in Microeconomic Theory.* Princeton University Press, Princeton, NJ, 1990.

[12] Harry Markowitz. Portfolio selection. *Journal of Finance*, 7:77–99, 1952.

[13] J. W. Pratt. Risk aversion in the small and in the large. *Econometrica*, 32:122–136, 1964.

[14] M. Rudolf and W. T. Ziemba. Intertemporal surplus management. *Journal of Economic Dynamics and Control*, 28:975–990, 2004.

[15] Leonard J. Savage. *The Foundations of Statistics.* John Wiley and Sons, New York:, 1954. Rev. and en. ed. New York: Dover Publications, 1972.

[16] William F. Sharpe. Integrated asset allocation. *Financial Analysts Journal*, 43(5):25–32, Sep. - Oct. 1987.

[17] William F. Sharpe and Lawrence G. Tint. Liabilities — a new approach. *Journal of Portfolio Management*, 16(2):5–10, 1990.

[18] Suresh Sundaresan and Fernando Zapatero. Valuation, optimal asset allocation and retirement incentives of pension plans. *Review of Financial Studies*, 10(3):631–60, 1997.

[19] Jules H. van Binsbergen and Michael W. Brandt. Optimal asset allocation in asset liability management. National Bureau of Economic Research, Inc, NBER Working Papers: 12970, 2007.

[20] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188, 1977.

[21] Robert Wilson. The theory of syndicates. *Econometrica*, 36:119–132, 1968.

Table 1: Mean, standard deviation, and asset allocation applied to the Japanese public pension fund

	Expected Return	Standard Deviation	Portfolio Weight
Domestic Bond	3.00%	5.42%	67.00%
Domestic Stock	4.80%	22.27%	11.00%
Foreign Bond	3.50%	14.05%	8.00%
Foreign Stock	5.00%	20.45%	9.00%
Short-term Asset	2.00%	3.63%	5.00%

Source: `http://www.gpif.go.jp/kanri/pdf/kanri02iinkai192.pdf` (in Japanese)
`http://www.gpif.go.jp/kanri/pdf/kanri03_h20_p04.pdf` (in Japanese)

Table 2: Correlation parameters used by the Japanese public pension fund

	DB	DS	FB	FS	SA
Domestic Bond (DB)	1				
Domestic Stock (DS)	0.22	1			
Foreign Bond (FB)	-0.05	-0.29	1		
Foreign Stock (FS)	-0.01	0.25	0.55	1	
Short-term Asset (SA)	0.39	0.05	-0.03	-0.07	1

Source: Same as for Table 1

Table 3: Mean, standard deviation, and asset allocation in the three-asset case.

	Expected Return	Standard Deviation	Portfolio Weight
Bond	3.00%	5.42%	67.00%
Stock	4.50%	13.16%	28.00%
Riskless Asset	2.00%	3.63%	5.00%

Table 4: Correlation parameters used in the three-asset case

	Bond	Stock	Riskless Asset
Bond	1		
Stock	0.13	1	
Riskless Asset	0.39	-0.01	1

Table 5: Suboptimal portfolio strategy and inefficient risk-sharing rule

	RRT of old	RRT of young	CE of old	CE of young	CE of old (suboptimal)	CE of young (suboptimal)	Difference in total CE
Panel A. young:old = 1:3							
Case 1	0.01	0.01	115.87	38.62	115.87	6.05	32.57
Case 2	0.01	0.06	115.87	38.87	115.87	11.24	27.63
Case 3	0.01	0.17	115.87	39.4	115.87	28.32	11.09
Case 4	0.01	0.25	115.87	39.79	115.87	34.91	4.88
Case 5	0.01	0.46	115.87	40.83	115.87	40.16	0.68
Panel B. young:old = 1:2							
Case 1	0.01	0.01	102.99	51.5	102.99	20.31	31.18
Case 2	0.01	0.06	102.99	51.82	102.99	32.45	19.37
Case 3	0.01	0.17	102.99	52.54	102.99	47.88	4.66
Case 4	0.01	0.25	102.99	53.05	102.99	51.19	1.85
Case 5	0.01	0.46	102.99	54.44	102.99	54.38	0.06
Panel C. young:old = 1:1							
Case 1	0.01	0.01	77.24	77.24	77.24	48.57	28.67
Case 2	0.01	0.06	77.24	77.73	77.24	66.69	11.05
Case 3	0.01	0.17	77.24	78.81	77.24	77.27	1.54
Case 4	0.01	0.25	77.24	79.57	77.24	79.27	0.3
Case 5	0.01	0.46	77.24	81.66	77.24	81.3	0.36

Terminal horizon is $T = 5$ years. Initial aggregate wealth W_0 is 140 trillion Japanese yen. Initial wealth W_0 is redistributed to the young and old at a ratio of 1:3 (Panel A), 1:2 (Panel B), and 1:1 (Panel C). CE is the certainty equivalent of wealth (trillion Japanese yen) attained by the optimal portfolio strategy and an efficient risk-sharing rule. CE (suboptimal) is the certainty equivalent of wealth created by a suboptimal portfolio strategy and an inefficient risk-sharing rule. Young and old are homogeneous in Case 1, but heterogeneous in all other cases.