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Takanori Adachi

The Graduate School of International Corporate Strategy,
Hitotsubashi University

Ryozo Miura

The Graduate School of International Corporate Strategy,
Hitotsubashi University

Hidetoshi Nakagawa

The Graduate School of International Corporate Strategy,
Hitotsubashi University

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Credit Risk Modeling with Delayed Information [*][†]

Takanori Adachi [‡] Ryozo Miura [§] Hidetoshi Nakagawa [¶]

Abstract

We introduce a notion of market times that are stochastic processes in order to represent information delay in structural credit risk models. The market times are extensions of the time change process introduced by Guo, Jarrow and Zeng in the sense that each component of the market time is not required to be a stopping time. We introduce a class of market times called idempotent market times that contain natural examples including market times driven by Poisson processes. We show that any idempotent market time is hard to be a model of the time change process. We define a filtration modulated by the market time and show that it is an extension of the continuously delayed filtration that is the filtration modulated by the time change process. We show that the conditional expectations given market filtrations have some Markov property in a binomial setting, which is useful for pricing defaultable financial instruments.

Contents

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[‡] Graduate School of International Corporate Strategy, Hitotsubashi University Email: taka.adachi@gmail.com

[§] Graduate School of International Corporate Strategy, Hitotsubashi University Email: rmiura@ics.hit-u.ac.jp

[¶] Graduate School of International Corporate Strategy, Hitotsubashi University Email: hnakagawa@ics.hit-u.ac.jp

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1 Introduction

Starting with the trailblazing work of Merton [Mer74], a branch of credit risk modeling, called the structural approach has flourished by several authors (See e.g. Bielecki and Rutkowski [BR04] ,McNeil, Frey and Embrechts [MFE05] or Bielecki, Jeanblanc and Marek Rutkowski [BJR09]). Many of their models are defined so as to introduce components that make the model incomplete in the sense that its default time becomes a totally inaccessible stopping time.

The origin of this line is the work of Duffie and Lando [DL01]. They link the two perspectives by introducing noise into the market's information set. They postulate that the market can only observe the firm's asset value plus noise at equally spaced, discrete (non-continuous) time points. Kusuoka [Kus99] extends Duffie and Lando's model to continuous time observations. Nakagawa [Nak01] presents a filtering model of a default time in a rigid mathematical setting. Çetin, Jarrow, Protter and Yildirim [cJPY04] simply reduce the information the market can see instead of appending noise. Giesecke's model [Gie06] makes the default barrier be unobservable to the market.

Our approach that we present in this paper is toward the line. We focus on the market time delay as a source of the model incompleteness. Actually, there are earlier studies including the work by Lindset, Lund and Persson [LLP08] whose model has constant lags for both managers and markets, and more recently the work by Guo, Jarrow and Zeng [GJZ09] whose model is stochastic and is based on an increasing sequence of stopping times.

We enhance their approaches to the cases including not just deterministic delay but also some delay driven by (possibly non-stopping) random times. This enables us to consider a natural example of catching up to all information in a stochastically periodic manner that the continuous delayed model of Guo-Jarrow-Zeng does not accept.

The remainder of this paper consists of three sections.

In Section 2, we begin with an introduction of market times, showing the set of all market times forms a monoid. We introduce a family of market times, called idempotent market times whose member fails to be an example of the time change utilized by Guo, Jarrow and Zeng [GJZ09]. We provide a characterization of idempotent market times, We also give a couple of examples of market times in this section.

In Section 3, we present a definition of filtrations generated by market times, showing that they are natural extension of the continuously delayed filtrations of Guo, Jarrow and Zeng [GJZ09] in the sense that they coincide each other when the market time consists of stopping times.

In Section 4, we investigate behavior of market times in a binomial model. We show a conditional expectation given a market filtration has a strong Markov property.

In Section 5, we provide a valuation of a defaultable bond and its credit spread based on the calculation of a conditional expectation given a market filtration.

2 Market Times

Let $\mathcal{T}$ be a fixed time domain that has the least element 0, equipped with an adequate topology, such as $\{0, \delta, 2\delta, \ldots, N\delta\}$ ($\delta > 0$), $[0, T]$ or $\mathbb{R}_+ := [0, \infty)$. For discrete domains, we assume that their topologies are discrete, that is, the powersets of the domains. For subsets of $\mathbb{R}$, their topologies consist of usual open sets. We sometimes see the time domain as a measurable space whose σ-field are generated by its open sets.

Definition 2.1. Let $\mathcal{T}$ be a time domain.

1. For $s, t \in \mathcal{T}$, $[s, t]_{\mathcal{T}} := \{u \in \mathcal{T} \mid s \leq u \leq t\}$. Similarly, we define $[s, t[_{\mathcal{T}}$, $]s, t]_{\mathcal{T}}$ and $]s, t[_{\mathcal{T}}$,
2. $\mathcal{T}_+ := \mathcal{T} - \{0\}$,
3. For a function f whose domain is $\mathcal{T}$, $f(t-) := \lim_{s \to t-0} f(s)$ and $f(t+) := \lim_{s \to t+0} f(s)$.

Note that in the case $\mathcal{T} = \{n\delta \mid n = 0, 1, \ldots\}$, $t- = t - \delta$ for $t \in \mathcal{T}_+$ and $t+ = t + \delta$ for $t \in \mathcal{T}$.

In this paper, all the discussion is under the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathcal{T}}, \mathbb{P})$, where the filtration $\mathbb{F}$ satisfies the usual condition for continuous time domains.

2.1 The Space of Market Times

Definition 2.2. [Random Times]

A *random time* is a random variable τ whose codomain is $\mathcal{T}$, that is, $\tau : \Omega \to \mathcal{T}$.

We denote the set of all random times by $\mathcal{T}^*$.

Definition 2.3. Let $p = p(\omega, t_1, t_2, \ldots, t_n)$ be a proposition on $\Omega \times \mathcal{T}^n$. Define predicates $AS(p)$, $SS(p)$, $IS(p)$ and $CS(p)$ by

$$
\begin{aligned}
AS(p) &:= \big((\forall t_1 \in \mathcal{T})(\forall t_2 \in \mathcal{T}) \ldots (\forall t_n \in \mathcal{T})\, \mathbb{P}\{\omega \in \Omega \mid p(\omega, t_1, t_2, \ldots, t_n)\} = 1\big), \\
SS(p) &:= \big((\forall \tau_1 \in \mathcal{T}^*)(\forall \tau_2 \in \mathcal{T}^*) \ldots (\forall \tau_n \in \mathcal{T}^*)\, \mathbb{P}\{\omega \in \Omega \mid p(\omega, \tau_1(\omega), \tau_2(\omega), \ldots, \tau_n(\omega))\} = 1\big), \\
IS(p) &:= \big(\mathbb{P}\{\omega \in \Omega \mid (\forall t_1 \in \mathcal{T})(\forall t_2 \in \mathcal{T}) \ldots (\forall t_n \in \mathcal{T}) p(\omega, t_1, t_2, \ldots, t_n)\} = 1\big), \\
CS(p) &:= \big((\forall t_1 \in \mathcal{T})(\forall t_2 \in \mathcal{T}) \ldots (\forall t_n \in \mathcal{T})(\forall \omega \in \Omega) p(\omega, t_1, t_2, \ldots, t_n)\big).
\end{aligned}
$$

Theorem 2.4. *Let p be a proposition on $\Omega \times \mathcal{T}^n$. Then, we have the following implications.*

$$
\begin{array}{ccccc}
 & & SS(p) & & \\
 & \nearrow & & \searrow & \\
CS(p) & & & & AS(p) \\
 & \searrow & & \nearrow & \\
 & & IS(p) & &
\end{array}
$$

Proof. Immediate. □

Definition 2.5. Let X and Y be two stochastic processes.

1. Y is called a *modification* of X if $AS(X_t(\omega) = Y_t(\omega))$,
2. Y is called a *strong modification* of X if $SS(X_t(\omega) = Y_t(\omega))$.
3. X and Y are called *indistinguishable* if $IS(X_t(\omega) = Y_t(\omega))$.

In the following, we sometimes drop the occurrences of ω in arguments of predicate constructors like $SS(X_t = Y_t)$ instead of writing $SS(X_t(\omega) = Y_t(\omega))$.

Now we have a definition of the key concept of this paper.

Definition 2.6. [Market Times]

A *raw market time* is a stochastic process $m : \mathcal{T} \times \Omega \to \mathcal{T}$ satisfying the following conditions,

1. $SS(m_0 = 0)$,
2. $SS(m_t \leq t)$,
3. $SS(t_1 \leq t_2 \ \to \ m_{t_1} \leq m_{t_2})$.

An $\mathbb{F}$-*market time* is a raw market time which is $\mathbb{F}$-adapted.

The market time m_t represents the delayed time in the sense that if the market knows an event at time t, then the event actually happened at time m_t (ahead of t) when managers learned it. So, it models the fact that the market will know the information (slightly) after the managers know it, that is, representing asymmetric information.

Lindset et al. introduced the two time lags for markets and managers in [LLP08]. Their lags are constant and not stochastically varying like ours.

Proposition 2.7. *Let m be a raw market time and m' be a strong modification of m. Then, m' is a raw market time.*

Proof. Straightforward. □

Proposition 2.7 asserts that we can treat the space of market times as a quotient space safely.

Definition 2.8. [Space of Market Times]

1. $\mathcal{M}$ is the set of all raw market times.
2. For $m^1, m^2 \in \mathcal{M}$, the *composite process* $m^1 \circ m^2$ is defined by for $t \in \mathcal{T}$ and $\omega \in \Omega$,
$$(m^1 \circ m^2)_t(\omega) = (m^1 \circ m^2)(t, \omega) := m^1(m^2(t, \omega), \omega).$$
3. $\mathbb{M} := \mathcal{M} / \sim$, where $\sim$ is a binary relation on $\mathcal{M}$ defined by $m^1 \sim m^2$ iff m^1 is a strong modification of m^2 for any pair of m^1 and m^2 in $\mathcal{M}$,
   For $m \in \mathcal{M}$, we write $m \in \mathbb{M}$ by identifying m with the equivalence class $[m]_\sim \in \mathbb{M}$ if it leads no confusion.
4. An *identity process* is a process $\mathbb{1}^{\mathcal{M}} \in \mathcal{M}$ defined by $\mathbb{1}^{\mathcal{M}}_t(\omega) = t$ for all $t \in \mathcal{T}$ and $\omega \in \Omega$.

Theorem 2.9. *The structure $\langle \mathbb{M}, \circ, \mathbb{1}^{\mathcal{M}} \rangle$ forms a monoid* [1]*, where $\circ$ is a well-defined operator on $\mathbb{M}$ induced by the operator $\circ$ on $\mathcal{M}$.*

Proof. First, we show that the process $(m^1 \circ m^2)$ is a market time, that is, the operation $\circ$ is well-defined and the set $\mathcal{M}$ is closed under the operation $\circ$, by examining if it satisfies the three condition in Definition 2.6, which is actually straightforward.

[1] A semigroup with identity.

Next, we check the associativity of the operator $\circ$. Let $m^1, m^2, m^3 \in \mathcal{M}$. We have for any $t \in \mathcal{T}$ and $\omega \in \Omega$,

$$((m^1 \circ m^2) \circ m^3)_t(\omega) = m^1_{m^2_{m^3_t(\omega)}(\omega)}(\omega) = (m^1 \circ (m^2 \circ m^3))_t(\omega).$$

The last thing we have to check is that $\mathbb{1}^{\mathcal{M}}$ is an identity of the operator $\circ$. But, this is also straightforward. $\square$

2.2 Idempotent Market Times

Definition 2.10. [Idempotent Market Times] A raw market time m is called *idempotent* if

$$SS(m_{m_t} = m_t), \tag{1}$$

or $m \circ m$ is a strong modification of m.

Proposition 2.11. *A raw market time m is idempotent iff* $SS(m_{t_1} \le t_2 \le t_1 \ \rightarrow \ m_{t_1} = m_{t_2})$.

Proof. If part. For $\tau \in \mathcal{T}^*$, we have

$$\{m_\tau \le \tau\} \cap \{m_\tau \le m_\tau \le \tau \ \rightarrow \ m_\tau = m_{m_\tau}\} \subset \{m_\tau = m_{m_\tau}\}.$$

By the assumption, the probability of the left hand set is 1. Therefore, $\mathbb{P}\{m_\tau = m_{m_\tau}\} = 1$ as well.

Only if part. For any $\tau_1, \tau_2 \in \mathcal{T}^*$, define a set A by

$$A := \{m_{m_{\tau_1}} = m_{\tau_1}\} \cap \{m_{\tau_1} \le \tau_2 \ \rightarrow \ m_{m_{\tau_1}} \le m_{\tau_2}\} \cap \{\tau_2 \le \tau_1 \ \rightarrow \ m_{\tau_2} \le m_{\tau_1}\}.$$

Then, we have $\mathbb{P}(A) = 1$ since m is an idempotent raw market time. Now, observing

$$\begin{aligned}
&A \cap \{m_{\tau_1} \le \tau_2 \le \tau_1\} \\
=&A \cap \{m_{\tau_1} \le \tau_2\} \cap \{\tau_2 \le \tau_1\} \\
\subset&\{m_{m_{\tau_1}} = m_{\tau_1}\} \cap \{m_{m_{\tau_1}} \le m_{\tau_2}\} \cap \{m_{\tau_2} \le m_{\tau_1}\} \\
=&\{m_{m_{\tau_1}} = m_{\tau_1}\} \cap \{m_{m_{\tau_1}} \le m_{\tau_2} \le m_{\tau_1}\} \\
\subset&\{m_{\tau_1} = m_{\tau_2}\},
\end{aligned}$$

we have $A \subset \{m_{\tau_1} \le \tau_2 \le \tau_1 \ \rightarrow \ m_{\tau_1} = m_{\tau_2}\}$. Therefore, $\mathbb{P}\{m_{\tau_1} \le \tau_2 \le \tau_1 \ \rightarrow \ m_{\tau_1} = m_{\tau_2}\} = 1$. $\square$

Here is one of the important implications derived from Proposition 2.11.

Corollary 2.12. *Let $m = \{m_t\}_{t \in \mathcal{T}}$ be an idempotent $\mathbb{F}$-market time where each m_t is a $\mathbb{F}$-stopping time. Then, for every pair t and s in $\mathcal{T}$ with $t \ge s$, we have $\{m_t = m_s\} \in \mathcal{F}_s$.*

Proof. Let $A \subset \Omega$ be the set defined by $A := \{m_t \le s \le t \ \rightarrow \ m_t = m_s\} \cap \{m_s \le s\}$. Then, since m is a market time and by Proposition 2.11, we get $\mathbb{P}(A) = 1$.

Now, under the assumption $s \le t$, we have

$$A \cap \{m_t \le s\} = A \cap (\{m_t \le s \le t \ \rightarrow \ m_t = m_s\} \cap \{m_t \le s\}) \subset A \cap \{m_t = m_s\}$$

and

$$A \cap \{m_t = m_s\} = A \cap (\{m_s \le s\} \cap \{m_t = m_s\}) \subset A \cap \{m_t \le s\}.$$

Thus $A \cap \{m_t \le s\} = A \cap \{m_t = m_s\}$. Therefore $\{m_t \le s\} \triangle \{m_t = m_s\} \subset \Omega - A$. Hence $\mathbb{P}(\{m_t \le s\} \triangle \{m_t = m_s\}) = 0$. Since $\{m_t \le s\}$ is $\mathcal{F}_s$-measurable and $\mathcal{F}_s$ is complete, we have $\{m_t = m_s\} \in \mathcal{F}_s$. $\square$

Let us think s and t to be the current time and any future time, respectively. Then by Corollary 2.12, we can know if the information will have increased since now by *any future* time t, which is not realistic.

So, we should conclude that requiring each random time m_t to be a stopping time is not practical in the case that m is idempotent while some of the idempotent market times are quite interesting both in the practical and the theoretical sense. This is our original motivation to develop a delayed theory that does not depend on stopping times.

The following theorem gives an insight about the shape of idempotent market times.

Theorem 2.13. *Let $m : \mathcal{T} \times \Omega \to \mathcal{T}$ be an idempotent raw market time. Then, we have*

$$AS(m_t = t \ \vee \ m_t = m_{t-}).$$

Proof. For any $t \in \mathcal{T}$, make a following calculation:

$$\begin{aligned}
A &:= \{m_t \leq t\} \cap \{m_{m_t} = m_t\} \cap \{m_t \leq t- \ \to \ m_{m_t} \leq m_{t-}\} \cap \{t- \leq t \ \to \ m_{t-} \leq m_t\} \\
&\subset \{m_t \leq t\} \cap \{m_t \leq t- \ \to \ m_t \leq m_{t-}\} \cap \{t- \leq t \ \to \ m_{t-} \leq m_t\} \\
&= \{m_t \leq t\} \cap \{m_t < t \ \to \ m_t \leq m_{t-}\} \cap \{m_{t-} \leq m_t\} \\
&= (\{m_t = t\} \cup \{m_t < t\}) \cap \{m_t < t \ \to \ m_t \leq m_{t-}\} \cap \{m_{t-} \leq m_t\} \\
&\subset (\{m_t = t\} \cap \{m_{t-} \leq m_t\}) \cup (\{m_t \leq m_{t-}\} \cap \{m_{t-} \leq m_t\}) \\
&\subset \{m_t = t\} \cup \{m_t = m_{t-}\} \\
&= \{m_t = t \ \vee \ m_t = m_{t-}\}.
\end{aligned}$$

Here, $\mathbb{P}(A) = 1$ since m is an idempotent raw market time.

Therefore, we have $\mathbb{P}\{m_t = t \ \vee \ m_t = m_{t-}\} = 1$.

□

Next, we show a characterization of idempotent market times.

Definition 2.14. 1. For a random set $M \subset \mathcal{T} \times \Omega$, define a process $m^M : \mathcal{T} \times \Omega \to \mathcal{T}$ by

$$m^M_t(\omega) = \sup\{s \leq t \mid (s, \omega) \in M\}, \tag{2}$$

where we use the convention $\sup \emptyset = 0$.

2. For a raw market time m, define a random set M^m by

$$M^m := \{(t, \omega) \in \mathcal{T} \times \Omega \mid m_t(\omega) = t\}. \tag{3}$$

Note that m^M_t is the end [2] of the random set $M_t := M \cap ([0, t]_{\mathcal{T}} \times \Omega)$.

Proposition 2.15. *1. Let $M \subset \mathcal{T} \times \Omega$ be a random set. Then, the process m^M is an idempotent raw market time.*

2. Let m be an idempotent raw market time, Then, the process m^{M^m} is a strong modification of m.

Proof. 1. It is clear that m^M is a raw market time. So, let us show it is also idempotent. Let $\omega \in \Omega$, $\tau \in \mathcal{T}^*$ and $s := m^M_\tau(\omega)$. Then,

$$s \leq \tau(\omega) \quad \text{and} \quad (\forall u \in \mathcal{T}) u \leq \tau(\omega) \wedge (u, \omega) \in M \to u \leq s. \tag{4}$$

Now, it is enough to show that $\{u \leq s \mid (u, \omega) \in M\} = \{u \leq \tau(\omega) \mid (u, \omega) \in M\}$. Since $s \leq \tau(\omega)$, it is obvious that **LHS** $\subset$ **RHS**. Let $u \in$ **RHS**. Then by Equation (4), $u \leq s$. Therefore, $u \in$ **LHS**.

[2] See Definition 2.23.

2. For $\omega \in \Omega$, $s \in \mathcal{T}$ and $\tau \in \mathcal{T}^*$, We have

$$m_\tau^{M^m}(\omega) = \sup\{s \le \tau(\omega) \mid (s,\omega) \in M^m\} = \sup\{s \in \mathcal{T} \mid s \le \tau(\omega) \wedge m_s(\omega) = s\}. \tag{5}$$

Then by Equation (5), we have $\{m_\tau \le \tau\} \cap \{m_{m_\tau} = m_\tau\} \subset \{m_\tau^{M^m} \ge m_\tau\}$.

Here, the probability of the left hand set of the above equation is 1 since m is an idempotent market time. Therefore, $\mathbb{P}\{m_\tau^{M^m} \ge m_\tau\} = 1$.

On the other hand,

$$\begin{aligned}
&\{m_\tau < s \le \tau \ \rightarrow\ m_\tau = m_s\} \cap \{s \le \tau \ \wedge\ m_s = s\} \cap \{m_\tau < s\} \\
=&\{m_\tau < s \le \tau \ \rightarrow\ m_\tau = m_s\} \cap \{m_\tau < s \le \tau\} \cap \{m_\tau < s\} \cap \{m_s = s\} \\
\subset&\{m_\tau = m_s\} \cap \{m_\tau < s\} \cap \{m_s = s\} \\
\subset&\{m_s < s\} \cap \{m_s = s\} = \emptyset.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\{m_\tau < s \le \tau \ \rightarrow\ m_\tau = m_s\} &\subset \{(s \le \tau \ \wedge\ m_s = s) \ \rightarrow\ s \le m_\tau\} \\
&\subset \{m_\tau^{M^m} \le m_\tau\}.
\end{aligned} \tag{6}$$

Here the last inclusion holds by Equation (5). The probability of the left most statement of Equation (6) is 1 by Proposition 2.11. Therefore, $\mathbb{P}\{m_\tau^{M^m} \le m_\tau\} = 1$.

□

We have the following characterization theorem for idempotent raw market times.

Theorem 2.16. *Let $m : \mathcal{T} \times \Omega \to \mathcal{T}$ be a process. Then, m is an idempotent raw market time iff there exists a random set $M \subset \mathcal{T} \times \Omega$ such that m^M is a strong modification of m.*

Proof. Immediate from Proposition 2.15. □

Proposition 2.17. *Let M be an $\mathbb{F}$-progressive set. Then, the process m^M is an idempotent $\mathbb{F}$-market time.*

Proof. It is enough to show that m^M is $\mathbb{F}$-adapted. Since M is $\mathbb{F}$-progressive, $M_t = M \cap ([0,t]_{\mathcal{T}} \times \Omega)$ is $\mathcal{B}[0,t] \otimes \mathcal{F}_t$-measurable. Then, since m_t^M is the end of M_t, it is $\mathcal{F}_t$-measurable. □

Theorem 2.18. *Let $m : \mathcal{T} \times \Omega \to \mathcal{T}$ be a càdlàg process. Then, m is an idempotent $\mathbb{F}$-market time iff there exists an $\mathbb{F}$-optional set $M \subset \mathcal{T} \times \Omega$ such that m^M is a strong modification of m.*

Proof. If part. By Proposition 2.17 and the remark after Definition A.3.

Only if part. All we need to show is that the random set M^m defined by Equation (3) is $\mathbb{F}$-optional when m is $\mathbb{F}$-adapted.

For $n \in \mathbb{N}$, define processes p^n by $p^n := \mathbb{1}_{\{(t,\omega) | m_t(\omega) \le t < m_t(\omega) + \frac{1}{n}\}}$. Then, p^n is obviously $\mathbb{F}$-adapted and càdlàg. Therefore, $(p^n)^{-1}(1) = \{(t,\omega) \mid m_t(\omega) \le t < m_t(\omega) + \frac{1}{n}\}$ is an $\mathbb{F}$-optional set. Thus, so is $M^m = \bigcap_{n \in \mathbb{N}} (p^n)^{-1}(1)$. □

In the rest of this subsection, we only treat $\mathcal{T} = \mathbb{R}_+$ cases.

Theorem 2.19. *Let m be an idempotent raw market time whose sample paths are càdlàg. Define a generalized function X_t by*

$$X_t := \mathbb{1}_{I^m - J^m}(t) + \sum_{u \in J^m} (m_u - m_{u-})\delta(t-u) \tag{7}$$

where I^m and J^m are random sets defined by

$$I^m(\omega) := \{t > 0 \mid m_t(\omega) = t\} \quad \text{and} \quad J^m(\omega) := \partial I^m(\omega) = \{t \in I^m(\omega) \mid t- \notin I^m(\omega)\},$$

and δ is the Dirac delta function. Then, we have for every $t \in \mathbb{R}_+$,

$$AS\left(m_t = \int_0^t X_s ds\right). \tag{8}$$

Proof. By Theorem 2.13, we have $m_t \neq t \;\to\; m_t - m_{t-} = 0$. So, we have almost surely,

$$m_t = \mu\left(I^m \cap]0,t]\right) + \sum_{u \in J^m \cap]0,t]} (m_u - m_{u-}) \tag{9}$$

where μ is the Lebesgue measure. Noting that J^m is a countable set, Equation (8) is a straightforward consequence of Equation (9). □

Equation (7) suggests that we can write the class of idempotent market times in Queuing theoretical form like A/A/1, where A denotes the class of any positive random variables whose expectations are finite.

As the last topic of this subsection, we mention a family of market times called discretizors.

Definition 2.20. [Discretizors]

Let δ be a positive number. A *discretizor* with the resolution δ is a deterministic raw market time $\Delta^\delta = \{\Delta^\delta_t\}_{t \geq 0}$ defined by

$$\Delta^\delta_t = n\delta, \quad \text{where} \quad n \in \mathbb{N} \quad \text{with} \quad n\delta \leq t < (n+1)\delta.$$

This is an idempotent raw market time catching up with the managers' time every δ unit time. It also satisfies $t - \delta < \Delta^\delta_t \leq t$ for every $t \geq 0$.

Discretizors are sometimes used for making a given market time that has a continuous distribution be its approximate market time with a discrete distribution. The following proposition gives a basic principle of the approximation.

Proposition 2.21. *Let $m \in \mathcal{M}$, and $t \geq 0$.*

1. $\lim_{\delta \to 0} \Delta^\delta_t = \mathbb{1}^{\mathcal{M}}_t$.
2. $(\Delta^\delta \circ m)_t \to m_t$ *pointwise on Ω as $\delta \to 0$.*
3. $(m \circ \Delta^\delta)_t \to m_t$ *pointwise on Ω as $\delta \to 0$ if every sample path of the original market time m_t is left-continuous.*

Proof. 1. Immediate from the inequation $t - \delta < \Delta^\delta_t \leq t$.

2. Immediate from the inequation $m_t(\omega) - \delta < \Delta^\delta_{m_t(\omega)} \leq m_t(\omega)$.

3. Noticing that Δ^δ_t approaching to t from left, $(m \circ \Delta^\delta)_t$ converges to m_t if m_t is left-continuous.

□

Note that the cardinality of the range of the market time $\Delta^\delta \circ m$ is at most countable.

2.3 Honest Times

In Proposition 2.12, we were discouraged to make a market time consist of stopping times when it is idempotent.

In this subsection, we revisit the issue by adopting a wider class of random times than the class of stopping times.

Definition 2.22. [Honest Times]

A random time τ is called $\mathbb{F}$-*honest* with respect to a $\mathbb{F}$-adapted process $\{\tau_t\}_{t\in\mathcal{T}_+}$ on $\mathcal{T}$ if for every $t \in \mathcal{T}_+$, $\tau = \tau_t$ on $\{\tau \leq t\}$, i.e. $\tau \mathbb{1}_{\{\tau\leq t\}} = \tau_t \mathbb{1}_{\{\tau\leq t\}}$. A random time τ is called $\mathbb{F}$-*honest* if there exists a $\mathbb{F}$-adapted process $\{\tau_t\}_{t\in\mathcal{T}_+}$ such that τ is $\mathbb{F}$-honest with respect to $\{\tau_t\}_{t\in\mathcal{T}_+}$.

It is well known that every $\mathbb{F}$-stopping time is $\mathbb{F}$-honest (See e.g. page 373 of Protter [Pro04] or page 384 of Nikeghbali [Nik06]).

Here is a also a well known characterization of honest times by optional processes.

Definition 2.23. Let $A \subset \mathcal{T} \times \Omega$ be a random set. The *end* of A is the random time E_A defined by

$$E_A(\omega) := \sup\{t \in \mathcal{T} \mid (t,\omega) \in A\}, \tag{10}$$

where we use the convention $\sup \emptyset = 0$.

Theorem 2.24. *[[Pro04] Theorem VI.16] A random time τ is $\mathbb{F}$-honest if and only if there exists an $\mathbb{F}$-optional set A such that $\tau = E_A$.*

The following is a very nice characterization of honest times developed by Yor (Yor [Yor78]).

Theorem 2.25. *[Yor [Yor78]] A random time τ is $\mathbb{F}$-honest if and only if for every $u \in [0, s[_{\mathcal{T}}$,*

$$(\exists A \in \mathcal{F}_s)\{\tau \leq u\} = A \cap \{\tau \leq s\}. \tag{11}$$

Our first question in this subsection is for a given market time $m = \{m_t\}_{t\in\mathcal{T}}$, if there exists an honest time τ with respect to m.

Here is a necessary and sufficient condition of the existence of such τ.

Proposition 2.26. *Let $m = \{m_t\}_{t\in\mathcal{T}}$ be an $\mathbb{F}$-market time. Then, a random time $\tau : \Omega \to \mathcal{T} \cup \{\infty\}$ is $\mathbb{F}$-honest with respect to m if and only if for every $\omega \in \Omega$,*

$$m_\infty(\omega) := \lim_{t\to\infty} m_t(\omega) = \tau(\omega) = m_{\tau(\omega)}(\omega).$$

Proof. Note that the random time τ is $\mathbb{F}$-honest with respect to m iff

$$(\forall t \in \mathcal{T}_+)(\forall \omega \in \Omega)\big[\tau(\omega) \leq t \to m_t(\omega) = \tau(\omega)\big]. \tag{12}$$

Only if part: Since m_t is monotonic, $\lim_{t\to\infty} m_t(\omega) = \sup_{t\in\mathcal{T}} m_t(\omega)$. Therefore, the result comes immediately by Equation (12).

If part: Since $\sup_{t\in\mathcal{T}} m_t(\omega) = \tau(\omega)$,

$$(\forall t \in \mathcal{T}_+) m_t(\omega) \leq \tau(\omega).$$

So, it is sufficient to show $m_t(\omega) \geq \tau(\omega)$, assuming $\tau(\omega) \leq t$. But, by the monotonicity of m_t and the assumption $\tau(\omega) = m_{\tau(\omega)}(\omega)$, we have

$$\tau(\omega) = m_{\tau(\omega)}(\omega) \leq m_t(\omega).$$

□

As an implication of Proposition 2.26, we missed the possibility of making whole market time be characterized by one honest time if the market time is unbounded. However, we have the following fairly nice theorem of asserting each m_t becomes an honest time for some market times including Poisson market times.

Theorem 2.27. *If $m = \{m_t\}_{t\in\mathcal{T}}$ is an idempotent $\mathbb{F}$-market time, then for every $t \in \mathcal{T}$, m_t is an $\mathbb{F}$-honest time.*

Proof. Define a random field $\{\tau_s^t\}_{t,s\in\mathcal{T}}$ by $\tau_s^t := m_{t\wedge s}$.

Then, it is obvious that τ_s^t is $\mathcal{F}_s$-measurable. So, all we need to show is $\tau_s^t = m_t$ on $\{m_t \leq s\}$.

If $s \geq t$, we have $\tau_s^t = m_t$ on Ω. Hence, we concentrate on the case $s < t$. Now for any $\omega \in \{m_t \leq s\}$, $m_t(\omega) \leq s < t$. Then, since m is idempotent, we get $m_t(\omega) = m_{m_t(\omega)}(\omega) \leq m_s(\omega) \leq m_t(\omega)$. Therefore, $m_t(\omega) = m_s(\omega) = \tau_s^t(\omega)$. □

Here is another characterization of honest times by using idempotent market times.

Theorem 2.28. *A random time $\tau : \Omega \to \bar{\mathcal{T}}$ is $\mathbb{F}$-honest if and only if there exists an idempotent $\mathbb{F}$-market time m such that for every $t \in \mathcal{T}_+$, $\tau = m_t$ on $\{\tau \leq t\}$, i.e. $\tau = m_\infty$.*

Proof. If part. Immediate by Definition 2.22.

Only if part. By Theorem 2.24, there exists an $\mathbb{F}$-optional set M such that $\tau = E_M$ since τ is $\mathbb{F}$-honest.

Let $m := m^M$. Then, by Theorem 2.18, m is an idempotent $\mathbb{F}$-market time.

On the other hand, for $\omega \in \{\tau \leq t\}$, we have

$$\begin{aligned} m_t(\omega) &= \sup\{s \leq t \mid (s,\omega) \in M\} \\ &= \sup\{s \in \mathcal{T} \mid (s,\omega) \in M\} \quad \text{since} \quad s \leq \tau(\omega) \leq t \\ &= E_M(\omega). \end{aligned}$$

Therefore, $m_t = E_M = \tau$. □

2.4 Examples of Market Times

We already see two sorts of concrete examples of market times, the identity market time and discretizors, which are both idempotent market times.

In this subsection we show more examples including stochastic market times. Among them, renewal market times and the starting times of Brownian excursions are idempotent market times. Therefore, they have difficulty to work as the time change processes that Guo, Jarrow and Zeng use when they define continuously delayed filtrations [GJZ09]. We will discuss the issue further in Section 2.4.2.

All examples are under $\mathcal{T} = \mathbb{R}_+$ case.

2.4.1 Constantly Delayed Market Times

The following is an example of deterministic market times taken from Lindset et al. [LLP08].

Definition 2.29. [Constantly Delayed Market Time] Let d be a positive constant. A raw market time $m = \{m_t\}_{t\in\mathcal{T}}$ is called a *constantly delayed market time* with delay d if for all $t \in \mathcal{T}$,

$$m_t := \max\{t - d, 0\}.$$

2.4.2 Renewal Market Times

The first example of the stochastic market times is the market time driven by independent identically distributed interval times, which, consequently, has multiple jumps.

Definition 2.30. [Renewal Market Time]

1. $X_n \sim$ i.i.d. random variables such that $0 < \mathbb{E}^{\mathbb{P}}[X_n] < \infty$ for $n = 1, 2, \ldots,$
2. $S_n := \sum_{k=1}^{n} X_k,$
3. $N_t := \sup\{n \mid S_n \leq t\},$
4. $m_t := S_{N_t}.$

Intuitively, the random variable X_n specifies an interval time between n-th and $n+1$-th jumps when the market time catch up with the managers' time. The process N_t is the renewal process generated by those X_n's.

The renewal market time can be seen as a stochastic version of a discretizor where its resolution time changes statistically. It also can be written in Queuing theoretical form like G/0/1, where G stands for the class of general iid random variables representing waiting times, 0 for 0-service time, and 1 for the number of service channels.

You can easily verify that the renewal market time is an idempotent market time. This means that the renewal market time fails to be an example of the time change process that Guo, Jarrow and Zeng use when they define a continuously delayed filtration [GJZ09].

If the random variables X_n above obey an exponential distribution $Exp(\lambda)$, this renewal market time is called a *Poisson market time* whose Queuing theoretical representation is M/0/1.

In reality, we see the situations satisfying Equation (1) occasionally including when the firm is under an audit activity by authorities, becoming all insider information available to the market.

Figure 1 shows a sample trajectory of a Poisson market time with $\lambda = 10$. We use this trajectory in Section 5.

2.4.3 Periodically Filled Market Times

The next example of the stochastic market times is another stochastic version of discretizors in the sense that its filled ratio against the managers' time is determined in a stochastic manner.

Definition 2.31. [Periodically Filled Market Time] Let δ be a fixed positive number.

1. $t_n := n\delta$
2. $m_0 := 0, \quad m_{t_n} := (1 - U_n)m_{t_{n-1}} + U_n t_n$ where $U_n \sim U[0,1]$ is an independently distributed uniform distribution.

2.4.4 Occupation Times

Here is a definition of occupation times taken from Example 6.2 in Chapter 3 of Karatzas and Shreve [KS98].

Definition 2.32. [Occupation Times]

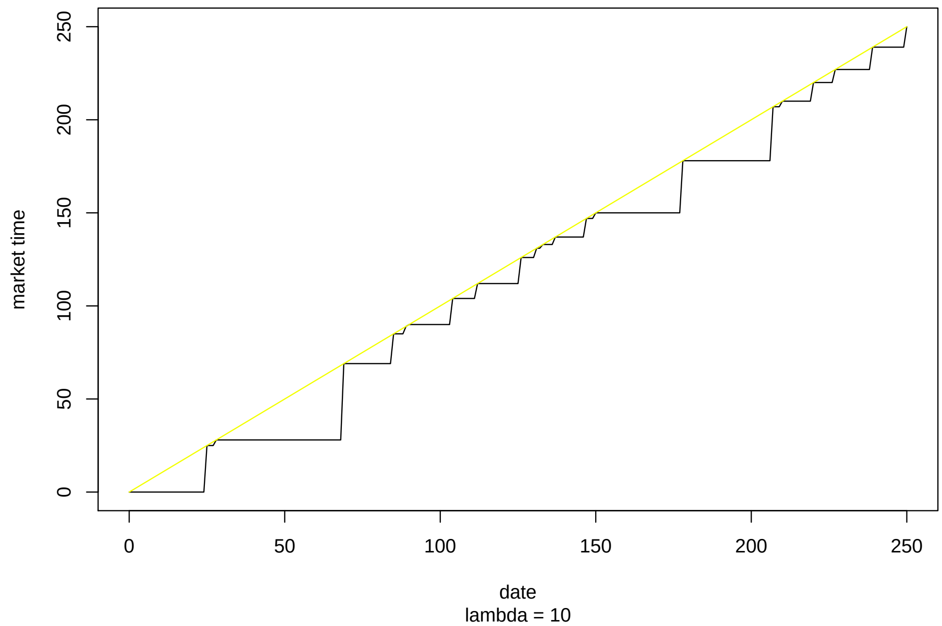

Figure 1: Poisson Market Time

Let $W = \{W_t\}_{t\in\mathcal{T}}$ be a Brownian motion and $B \in \mathcal{B}(\mathbb{R})$ be a Borel set. Then we define the *occupation time* of B by the Brownian path up to time t as

$$m_t := \int_0^t \mathbb{1}_B(W_s)ds.$$

Obviously, any occupation time m is a market time. However, the occupation time will not recover to the managers' time (that is, $m_t \neq t$) once it had a chance to walk out of the Borel set B. More precisely speaking, the delay $t - m_t$ is increasing as time passes, and never shrinks. Therefore, the converse is untrue.

Similarly, for a given continuous semimartingale $X = \{X_t\}_{t\in\mathcal{T}}$, its local time $L = \{L_t\}_{t\in\mathcal{T}}$ is a market time.

2.4.5 Starting Times for Excursions

Let $B = \{B_t\}_{t\in\mathcal{T}}$ be a standard Brownian motion, and define a random set Z by

$$Z = \{(t,\omega) \in \mathbb{R}_+ \times \Omega \mid B_t(\omega) = 0\}.$$

Then, the idempotent market time m^Z picks the starting times for the excursions out of 0 of B.

3 Market Filtrations

Definition 3.1. [Market Filtrations] Let $m = \{m_t\}_{t\in\mathcal{T}}$ be an $\mathbb{F}$-market time. The *market filtration* modulated by the $\mathbb{F}$-market time is the filtration $\mathbb{F}^m = \{\mathcal{F}_t^m\}_{t\in\mathcal{T}}$ defined by for $t \in \mathcal{T}$,

$$\mathcal{F}_t^m := \bigvee_{s\in[0,t]_\mathcal{T}} \mathcal{F}_{m_s}. \tag{13}$$

In Definition 3.1, $\mathcal{F}_{m_s}$ is the σ-field defined in Definition A.4.

Theorem 3.2. *Let $m = \{m_t\}_{t\in\mathcal{T}}$ be an $\mathbb{F}$-market time. Then the market filtration $\mathbb{F}^m$ is a subfiltration of $\mathbb{F}$.*

Proof. It is obvious that $\mathbb{F}^m$ is a filtration. So all we need to show is that $\mathcal{F}_t^m \subset \mathcal{F}_t$ for any $t \in \mathcal{T}$. But for any $s \le t$, since $m_s \le m_t \le t$, we have $\mathcal{F}_{m_s} \subset \mathcal{F}_t$ by Theorem A.6. Therefore, $\mathcal{F}_t^m = \bigvee_{s\in[0,t]_\mathcal{T}} \mathcal{F}_{m_s} \subset \mathcal{F}_t$.

□

The following theorem shows that our market filtration is a natural extension of the continuously delayed filtration of Guo, Jarrow and Zeng [GJZ09].

Theorem 3.3. *Let $m = \{m_t\}_{t\in\mathcal{T}}$ be an $\mathbb{F}$-market time where each m_t is an $\mathbb{F}$-stopping time. Then, $\mathcal{F}_t^m = \mathcal{F}_{m_t}$.*

Proof. Let $s, t \in \mathcal{T}$ with $s \le t$. Then $m_s \le m_t$.

First, we want to show $\mathcal{F}_{m_s} \subset \mathcal{F}_{m_t}$. Let $A \in \mathcal{F}_{m_s}$. Then, by Theorem A.7, for any $u \in \mathcal{T}$, we have $A \cap \{m_s \le u\} \in \mathcal{F}_u$.

On the other hand, since $m_s \le m_t$, we have

$$A \cap \{m_t \le u\} = (A \cap \{m_s \le u\}) \cap \{m_t \le u\}$$

The first term of the right hand side belongs to $\mathcal{F}_u$ by the assumption, while the second term is also in $\mathcal{F}_u$ since m_t is an $\mathbb{F}$-stopping time. So again by Theorem A.7, we get $A \in \mathcal{F}_{m_t}$.

Then, we have $\mathcal{F}_t^m = \bigvee_{s\in[0,t]_\mathcal{T}} \mathcal{F}_{m_s} = \mathcal{F}_{m_t}$. □

Since a constant time is considered as a stopping time, we have the following corollary.

Corollary 3.4. *Assume that an $\mathbb{F}$-market time m is deterministic, i.e. there exists a deterministic function $f : \mathcal{T} \to \mathcal{T}$ such that for all $t \in \mathcal{T}$ and $\omega \in \Omega$, $m_t(\omega) = f(t)$. Then, we have for all $t \in \mathcal{T}$, $\mathcal{F}_t^m = \mathcal{F}_{f(t)}$.*

Next, we investigate the shape of market filtrations when the underlying market times are idempotent.

Lemma 3.5. *Let m be an idempotent $\mathbb{F}$-market time which is càdlàg. Then for every pair of $s, t \in \mathcal{T}$ with $s < t$, m_s is $\mathcal{F}_{m_t}$-measurable.*

Proof. Let $s \in \mathcal{T}$ and $B \in \mathcal{B}(\mathcal{T})$ be fixed. For any $n \in \mathbb{N}$, define processes p^n and $q^n : \mathcal{T} \times \Omega \to \mathbb{R}$ by

$$p^n := \mathbb{1}_{\{(u,\omega)\in\mathcal{T}\times\Omega | m_s(\omega)\in B,\ m_s(\omega)\le u < m_s(\omega)+\frac{1}{n}\}},$$
$$q^n := \mathbb{1}_{\{(u,\omega)\in\mathcal{T}\times\Omega | m_s(\omega)\in B,\ u \ge s+\frac{1}{n}\}}.$$

Then, since m_s is càdlàg, $\mathcal{F}_s$-adapted and $\mathcal{F}_{m_s}$-adapted by Proposition A.5, both p^n and q^n are $\mathbb{F}$-adapted and càdlàg. Therefore, $P^n, Q^n \in \mathcal{F}_{m_t}$ where

$$P^n := \left(p^n_{m_t}\right)^{-1}(1) = \{m_s \in B,\ m_s \le m_t < m_s + \frac{1}{n}\},$$
$$Q^n := \left(q^n_{m_t}\right)^{-1}(1) = \{m_s \in B,\ m_t \ge s + \frac{1}{n}\}.$$

Then, we have

$$P := \bigcap_{n\in\mathbb{N}} P^n = \{m_s \in B,\ m_s = m_t\} \in \mathcal{F}_{m_t},$$
$$Q := \bigcup_{n\in\mathbb{N}} Q^n = \{m_s \in B,\ m_t > s\} \in \mathcal{F}_{m_t}.$$

Therefore

$$P \cup Q = \{m_s \in B\} \cap \big(\{m_s = m_t\} \cup \{m_t > s\}\big) \in \mathcal{F}_{m_t}.$$

On the other hand, under the assumption $s \le t$, Proposition 2.11 implies that the two sets $\{m_t \le s\}$ and $\{m_s = m_t\}$ are identical by ignoring a null-measured difference. Hence

$$\{m_s \in B\} \cap \big(\{m_t \le s\} \cup \{m_t > s\}\big) = \{m_s \in B\} \in \mathcal{F}_{m_t}.$$

Therefore, m_s is $\mathcal{F}_{m_t}$-measurable. □

Theorem 3.6. *Let m be an idempotent $\mathbb{F}$-market time which is càdlàg. Then, for every $t \in \mathcal{T}$, we have $\mathcal{F}^m_t = \mathcal{F}_{m_t}$.*

Proof. Immediate by Lemma 3.5 and Theorem A.6. □

4 Market Times in a Binomial Model

In this section, we investigate some behavior of idempotent market times in a concrete binomial model. We will show a conditional expectation given a market filtration has a sort of strong Markov property.

4.1 The Setup

In this subsection, we define a binomial model.

Definition 4.1. [Time] Let δ be a given positive number.

1. $\mathcal{T} := \{n\delta \mid n = 0, 1, 2, \dots, N\}$,
2. The *horizon* is the number $T := N\delta$,
3. For $t \in \mathcal{T}_+$, $t- := t - \delta$,
4. For $t \in [0, T[_\mathcal{T}$, $t+ := t + \delta$.

Definition 4.2. [Measurable Space] In the following, $\mathfrak{H}$, $\mathfrak{T}$ and $\perp$ are distinct constants.

1. $\Omega := \{\mathfrak{H}, \mathfrak{T}\}^{\mathcal{T}_+}$.
   For $\omega \in \Omega$, we expand its domain to $\mathcal{T}$ by defining $\omega(0) := \perp$,
2. For $t \in \mathcal{T}$, the binary relation $\sim_t$ on Ω is defined by for $\omega, \omega' \in \Omega$,
   $$\omega \sim_t \omega' \;\leftrightarrow\; (\forall s \in]0, t]_\mathcal{T})\omega(s) = \omega'(s),$$
3. For $t \in \mathcal{T}$, $\mathcal{F}_t := \sigma(\Omega / \sim_t)$,
4. $\mathcal{F} := \mathcal{F}_T$.

We sometimes see the set Ω as a topological space equipped with the discrete topology. In other words, any subset of Ω is an open set.

Definition 4.3. [Probability] Let $p \in]0, 1[$ be a given number.

1. $\mathbb{P} : \Omega \to [0, 1]$ is defined by for $\omega \in \Omega$, $\mathbb{P}(\omega) := p^{\#\omega}(1-p)^{N-\#\omega}$ where $\#\omega$ is the cardinality of $\omega^{-1}(\mathfrak{H})$,

2. $\mathbb{P} : \mathcal{F} \to [0,1]$ is defined by for $A \in \mathcal{F}$, $\mathbb{P}(A) := \sum_{\omega \in A} \mathbb{P}(\omega)$.

Throughout the rest of this section, all discussions are under the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathbb{P})$ defined in Definitions 4.1, 4.2 and 4.3. We also fix a state space $(E, \mathcal{E})$ satisfying $(\forall x \in E)\{x\} \in \mathcal{E}$. Note that both $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(\mathcal{T}, 2^{\mathcal{T}})$ satisfy this condition.

Theorem 4.4. *A function $X : \Omega \to E$ is $\mathcal{F}_t$-measurable iff*

$$(\forall \omega \in \Omega)(\forall \omega' \in \Omega)\omega \sim_t \omega' \;\to\; X(\omega) = X(\omega').$$

Proof. Only if part. Let $\omega_0 \in \Omega$ and $x_0 := X(\omega_0)$. Then, we have $\omega_0 \in X^{-1}(x_0) \in \mathcal{F}_t$ since X is $\mathcal{F}_t$-measurable. On the other hand, by the definition of $\mathcal{F}_t$ we have $[\omega_0]_{\sim_t}$. Therefore, for any $\omega \in \Omega$ satisfying $\omega \sim_t \omega_0$, we have $\omega \in [\omega_0]_{\sim_t} \subset X^{-1}(x_0)$. Hence, $X(\omega) = x_0 = X(\omega_0)$.

If part. It is enough to show that $X^{-1}(x) \in \mathcal{F}_t$ for any $x \in E$. In case that $X^{-1}(x) = \emptyset$, it is trivial. So we can assume there exists $\omega \in \Omega$ such that $X(\omega) = x$. Then for every $\omega' \in \Omega$ with $\omega' \sim_t \omega$, we have $X(\omega') = X(\omega) = x$ by the assumption. Hence, $\omega' \in X^{-1}(x)$. Therefore, $[\omega]_{\sim_t} \subset X^{-1}(x)$. Now since $X^{-1}(x)$ is finite, we have $X^{-1}(x) = \bigcup_{\omega \in X^{-1}(x)} [\omega]_{\sim_t} \in \mathcal{F}_t$. □

Corollary 4.5. *A process $Z : \mathcal{T} \times \Omega \to E$ is $\mathbb{F}$-adapted iff*

$$(\forall t \in \mathcal{T})(\forall \omega \in \Omega)(\forall \omega' \in \Omega)\omega \sim_t \omega' \;\to\; Z(t,\omega) = Z(t,\omega').$$

Definition 4.6. [The Universal Process]

1. $\Omega^* := \cup_{t \in \mathcal{T}} \{\mathfrak{H}, \mathfrak{T}\}^{]0,t]_{\mathcal{T}}}$, where $\{\mathfrak{H}, \mathfrak{T}\}^{\emptyset} := \{\perp\}$.
2. For $\omega \in \Omega$ and $t \in \mathcal{T}$, a function $\omega|t \in \{\mathfrak{H}, \mathfrak{T}\}^{]0,t]_{\mathcal{T}}}$ is defined by $\omega|t := \omega|_{]0,t]_{\mathcal{T}}}$ whose domain is expanded to $[0,t]_{\mathcal{T}}$ by defining $(\omega|t)(0) := \perp$,
3. The *universal process* is a process $\pi : \mathcal{T} \times \Omega \to \Omega^*$ defined by $\pi(t,\omega) := \omega|t$.

The following theorem shows that the universal process has a so-called universal property.

Theorem 4.7. *Let $Z : \mathcal{T} \times \Omega \to E$ be any $\mathbb{F}$-adapted process.*

1. *There exists a unique function $f : \Omega^* \to E$ such that $Z = f \circ \pi$,*
2. *For any $t \in \mathcal{T}$, $\sigma(Z_t) \subset \sigma(\pi_t)$.*

Proof. 1. For $\omega_0 \in \{\mathfrak{H}, \mathfrak{T}\}^{]0,t]_{\mathcal{T}}}$, define $f(\omega_0)$ by $f(\omega_0) := Z(t,\omega)$ where $\omega \in \Omega$ is the function defined by

$$\omega(s) := \begin{cases} \omega_0(s) & \text{if } s \in]0,t]_{\mathcal{T}}, \\ \mathfrak{H} & \text{otherwise.} \end{cases}$$

Then, we have $(f \circ \pi)(t,\omega) = f(\omega|t) = Z(t,\omega')$ where

$$\omega'(s) := \begin{cases} \omega(s) & \text{if } s \in]0,t]_{\mathcal{T}}, \\ \mathfrak{H} & \text{otherwise.} \end{cases}$$

Since $\omega' \sim_t \omega$, we have $Z(t,\omega) = Z(t,\omega')$ by Corollary 4.5.
Next we show the uniqueness. Suppose there is another function $f' : \Omega^* \to E$ such that $Z = f' \circ \pi$. Then, for any $\omega \in \Omega$ and $t \in \mathcal{T}$, we have $f'(\omega|t) = f(\omega|t)$. Since all elements of Ω^* can be represented as $\omega|t$ for some $\omega \in \Omega$ and $t \in \mathcal{T}$, the rest is straightforward.

2. By 1, we have $f : \Omega^* \to E$ such that $Z = f \circ \pi$. Then, for any $A \in \mathcal{E}$,

$$Z_t^{-1}(A) = \{\omega \in \Omega \mid Z(t,\omega) \in A\} = \{\omega \in \Omega \mid \pi(t,\omega) \in f^{-1}(A)\} \in \sigma(\pi_t).$$

□

4.2 Market Filtrations in a Binomial Model

In the rest of this section, we assume that $m : \mathcal{T} \times \Omega \to \mathcal{T}$ is an arbitrary but fixed idempotent $\mathbb{F}$-market time.

Proposition 4.8. *$\mathcal{O}^{\mathbb{F}} = \sigma\{\{t\} \times [\omega]_{\sim_t} \mid t \in \mathcal{T}, \omega \in \Omega\}$, where $\mathcal{O}^{\mathbb{F}}$ is the optional σ-field defined in Definition A.3.*

Proof. Since Ω is equipped with the discrete topology, any function whose domain is Ω is continuous. Therefore,

$$\mathcal{O}^{\mathbb{F}} := \sigma\{Z \mid Z \text{ is an } \mathbb{F}\text{-adapted càdlàg process.}\} = \sigma\{Z \mid Z \text{ is an } \mathbb{F}\text{-adapted process.}\}.$$

Then by Theorem 4.7 (2), we have $\mathcal{O}^{\mathbb{F}} = \sigma(\pi)$ since π itself is $\mathbb{F}$-adapted.

Now remind that any element of Ω^* can be represented as $\omega|t$ for $\omega \in \Omega$ and $t \in \mathcal{T}_+$. Then, since

$$(\forall \omega \in \Omega)(\forall t \in \mathcal{T}_+)\pi^{-1}(\omega|t) = \{t\} \times [\omega]_{\sim_t},$$

we have the desired equation. ☐

Corollary 4.9. *A process $Z : \mathcal{T} \times \Omega \to E$ is $\mathbb{F}$-optional iff it is $\mathbb{F}$-adapted.*

Proposition 4.10. *$\mathcal{F}_t^m = \sigma(\pi_{m_t})$.*

Proof. By Theorem 3.6, Corollary 4.9 and Theorem 4.7 (2). ☐

Now we investigate the shape of the set $\pi_{m_t}^{-1}(x)$ for $x \in \Omega^*$ in order to characterize $\mathcal{F}_t^m$.

Definition 4.11. For a random time τ, a *neighborhood* of $\omega \in \Omega$ at τ is the set $N_\tau(\omega) := [\omega]_{\sim_{\tau(\omega)}}$.

Lemma 4.12. *For $\omega, \omega_0 \in \Omega$, $\omega \in N_{m_t}(\omega_0)$ implies $m_t(\omega) \geq m_t(\omega_0)$.*

Proof. Since m is $\mathbb{F}$-adapted and $\omega \sim_t \omega_0$,

$$m_{m_t(\omega_0)}(\omega) = m_{m_t(\omega_0)}(\omega_0) = m_t(\omega_0).$$

The right most equality holds because m is idempotent. On the other hand, we have $m_t(\omega_0) \leq t$. Therefore, $m_{m_t(\omega_0)}(\omega) \leq m_t(\omega)$. ☐

Definition 4.13. Let τ be a random time, and $\omega_0 \in \Omega$.

1. $K_\tau(\omega_0) := \{\omega \in N_\tau(\omega_0) \mid \tau(\omega) > \tau(\omega_0)\}$,
2. $\underline{K}_\tau(\omega_0) := \{\omega \in K_\tau(\omega_0) \mid (\forall \omega' \in K_\tau(\omega_0))(N_\tau(\omega) \subset N_\tau(\omega') \to N_\tau(\omega) = N_\tau(\omega'))\}$.

Proposition 4.14. *Let $t \in \mathcal{T}$, $\omega_0 \in \Omega$ and $x_0 := \pi_{m_t}(\omega_0)$. Then,*

$$\pi_{m_t}^{-1}(x_0) = N_{m_t}(\omega_0) - \cup\{N_{m_t}(\omega) \mid \omega \in \underline{K}_{m_t}(\omega_0)\}. \tag{14}$$

Proof. Let $\omega \in \pi_{m_t}^{-1}(x_0)$. Then, $\pi_{m_t}(\omega) = \pi_{m_t}(\omega_0)$. Thus, $\omega|m_t(\omega) = \omega_0|m_t(\omega_0)$. Therefore, $m_t(\omega) = m_t(\omega_0)$ and $\omega \sim_{m_t(\omega_0)} \omega_0$, which implies $\omega \in N_{m_t}(\omega_0)$.

Now, we show that $\omega' \in \underline{K}_{m_t}(\omega_0)$ implies $\omega \notin N_{m_t}(\omega')$. Since $\omega' \in K_{m_t}(\omega_0)$, we have $\omega' \in N_{m_t}(\omega_0)$ and $m_t(\omega') > m_t(\omega_0)$. Suppose $\omega \in N_{m_t}(\omega')$. Then by Lemma 4.12, $m_t(\omega) \geq m_t(\omega') > m_t(\omega_0)$, which contradicts to $m_t(\omega) = m_t(\omega_0)$. Therefore, we conclude $\omega \notin N_{m_t}(\omega')$ and $LHS \subset RHS$.

Next, we show the opposite inclusion. Let $\omega \in N_{m_t}(\omega_0) - \cup\{N_{m_t}(\omega) \mid \omega \in \underline{\mathrm{K}}_{m_t}(\omega_0)\}$. We want to show $\omega \in \pi_{m_t}^{-1}(x_0)$.

Since $\omega \in N_{m_t}(\omega_0)$, we have $m_t(\omega) \geq m_t(\omega_0)$ by Lemma 4.12. Suppose $m_t(\omega) > m_t(\omega_0)$. Then, $\omega \in K_{m_t}(\omega_0)$. We can pick $\omega' \in \underline{\mathrm{K}}_{m_t}(\omega_0)$ such that $N_{m_t}(\omega') \supset N_{m_t}(\omega)$. Therefore, $\omega \in N_{m_t}(\omega) \subset N_{m_t}(\omega')$. But this contradicts to the way of the selection of ω. Hence, we have $m_t(\omega) = m_t(\omega_0)$.

On the other hand, we have $\omega|m_t(\omega_0) = \omega_0|m_t(\omega_0)$ since $\omega \in N_{m_t}(\omega_0)$. Therefore,

$$\pi_{m_t}(\omega) = \omega|m_t(\omega) = \omega|m_t(\omega_0) = \omega_0|m_t(\omega_0) = \pi_{m_t}(\omega_0) = x_0.$$

□

Corollary 4.15. $\mathcal{F}_t^m = \sigma\{N_{m_t}(\omega) \mid \omega \in \Omega\}$.

4.3 Conditional Expectations Given a Market Filtration

We keep assuming that m is an idempotent $\mathbb{F}$-market time throughout this subsection.

Theorem 4.16. *Let Y be a random variable and X be an $\mathcal{F}_{m_t}$-measurable random variable. Then,*

$$\mathbb{E}^{\mathbb{P}}[Y \mid \mathcal{F}_t^m] = X \quad \textit{iff} \quad (\forall \omega_0 \in \Omega)(\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)}Y] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)}X]).$$

Proof. The only-if part is trivial. So we assume the right hand side. Let

$$\mathcal{G} := \{A \in \mathcal{F}_t^m \mid \int_A Y d\mathbb{P} = \int_A X d\mathbb{P}\}.$$

Then, all we need to show is $\mathcal{G} = \mathcal{F}_t^m$.

By the assumption, for any $\omega_0 \in \Omega$, we have $N_{m_t}(\omega_0) \in \mathcal{G}$. Now seeing Equation (14) and noticing that the following relations are satisfied for any $\omega_1, \omega_2 \in \underline{\mathrm{K}}_{m_t}(\omega_0)$,

1. $N_{m_t}(\omega_1) \subset N_{m_t}(\omega_0)$,
2. $N_{m_t}(\omega_1) = N_{m_t}(\omega_2)$ or $N_{m_t}(\omega_1) \cap N_{m_t}(\omega_2) = \emptyset$,

we have the following equation where all unions are disjoint-sum:

$$N_{m_t}(\omega_0) = \pi_{m_t}^{-1}(\pi_{m_t}(\omega_0)) \cup \left(\bigcup\{N_{m_t}(\omega) \mid \omega \in \underline{\mathrm{K}}_{m_t}(\omega_0)\}\right).$$

Therefore, by the assumption, we have

$$\mathcal{H} := \{\pi_{m_t}^{-1}(\pi_{m_t}(\omega_0)) \mid \omega_0 \in \Omega\} \subset \mathcal{G}.$$

Again, the elements of $\mathcal{H}$ are disjoint each other, and obviously $\cup\mathcal{H} = \Omega$. So, any element of $\mathcal{F}_t^m = \sigma(\pi_{m_t})$ can be represented as a disjoint sum of the elements of $\mathcal{H}$, which concludes $\mathcal{F}_t^m \subset \mathcal{G}$.

□

Now we specify a process Y and calculates its conditional expectation given the market filtration $\mathcal{F}_t^m$.

Definition 4.17. [Process Y]

1. For $t \in \mathcal{T}_+$, a Bernoulli process X is defined by

$$X_t(\omega) = \begin{cases} \sqrt{\delta} & \text{if } \omega(t) = \mathfrak{H} \\ -\sqrt{\delta} & \text{if } \omega(t) = \mathfrak{T}. \end{cases}$$

2. For $t \in \mathcal{T}$, a process M is defined by $M_t(\omega) := \sum_{s \in]0,t]_{\mathcal{T}}} X_s(\omega)$.

3. The process Y is defined by
$$Y_t(\omega) := y_0 + \nu t + \sigma M_t(\omega)$$
where y_0, ν and $\sigma \geq 0$ are constants.

Definition 4.18. Let $f : \mathbb{R} \to \mathbb{R}$ be a fixed function. Define a function $g : \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} g(0, y) &:= f(y), \\ g(t+, y) &:= p g(t, y + \nu\delta + \sigma\sqrt{\delta}) + (1-p) g(t, y + \nu\delta - \sigma\sqrt{\delta}). \end{aligned}$$

Theorem 4.19. *For any $s \geq t$,*
$$\mathbb{E}^{\mathbb{P}}[f(Y_s) \mid \mathcal{F}^m_t] = g(s - m_t, Y_{m_t}). \tag{15}$$

Proof. By Theorem 4.16, since $g(s - m_t, Y_{m_t})$ is $\mathcal{F}^m_t$-measurable, all we need to show is

$$(\forall \omega_0 \in \Omega)(\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} f(Y_s)] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} g(s - m_t, Y_{m_t})]).$$

Thinking about the shape of the set $N_{m_t}(\omega_0)$, we can prove it by showing

$$(\forall C \in \mathbb{R})(\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} f(Y_{m_t+u} + C)] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} g(u, Y_{m_t} + C)]) \tag{16}$$

by induction on $u \in [0, s - m_t]_{\mathcal{T}}$.

When $u = 0$, it is trivial. Assume Equation (16) holds at $u \in [0, s - m_t[_{\mathcal{T}}$. Then, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} f(Y_{m_t+u+} + C)] \\
=&\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} f(Y_{m_t+u+} + C) \mid \mathcal{F}_{m_t+u}]] \\
=&\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} f(Y_{m_t+u} + \nu\delta + \sigma X_{m_t+u+} + C) \mid \mathcal{F}_{m_t+u}]] \\
=&\mathbb{E}^{\mathbb{P}}[p\mathbb{1}_{N_{m_t}(\omega_0)} f(Y_{m_t+u} + \nu\delta + \sigma\sqrt{\delta} + C) + (1-p)\mathbb{1}_{N_{m_t}(\omega_0)} f(Y_{m_t+u} + \nu\delta - \sigma\sqrt{\delta} + C)] \\
=&p\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} f(Y_{m_t+u} + (\nu\delta + \sigma\sqrt{\delta} + C))] + (1-p)\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} f(Y_{m_t+u} + (\nu\delta - \sigma\sqrt{\delta} + C))] \\
=&p\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} g(u, Y_{m_t} + (\nu\delta + \sigma\sqrt{\delta} + C)] + (1-p)\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} g(u, Y_{m_t} + (\nu\delta - \sigma\sqrt{\delta} + C)] \\
=&\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} (p g(u, (Y_{m_t} + C) + \nu\delta + \sigma\sqrt{\delta}) + (1-p) g(u, (Y_{m_t} + C) + \nu\delta - \sigma\sqrt{\delta}))] \\
=&\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{m_t}(\omega_0)} g(u+, Y_{m_t} + C)].
\end{aligned}$$

Therefore, Equation (16) holds at $u+$ as well, which completes the proof. □

Corollary 4.20. *For any $s \geq t$,*
$$\mathbb{E}^{\mathbb{P}}[f(Y_s) \mid \mathcal{F}^m_t] = \mathbb{E}^{\mathbb{P}}[f(Y_s) \mid m_t, Y_{m_t}]. \tag{17}$$

Note 4.21. Practically, Corollary 4.20 is enough to price defaultable securities under a market-time based model since we can make it as accurate as possible by making δ smaller.

Suppose $p = \frac{1}{2}$. Then, as $\delta \to 0$, the process M converges to a standard Brownian motion in distribution by the Central Limit Theorem, and the process Y will satisfy

$$Y_t = y_0 + \nu t + \sigma B_t. \tag{18}$$

The function g defined in Definition 4.18 will be specified with an appropriate partial differential equation, and Equation (17) may hold at this continuous case.

We will use this insight when we calculate a defaultable bond price in Section 5.2.

5 Valuation of Defaultable Bonds

In this section, we apply the result of the theory of market times to a firm value process that is driven by a geometric Brownian motion. We assume that the probability measure $\mathbb{P}$ is a risk-neutral probability measure from now on. The notation in this subsection basically follows Example 3.1.1 of Bielecki and Rutkowski [BR04]. In this section. we only treat $\mathcal{T} := \mathbb{R}_+$ case.

5.1 The Setup

Assume that the firm value process $V = \{V_t\}_{t\in\mathcal{T}}$ satisfies the equation

$$dV_t = V_t\big((r-\kappa)dt + \sigma_V dB_t\big) \tag{19}$$

where $\kappa > 0$ represents the payout ratio, $\sigma_V > 0$ is the constant volatility coefficient, and B_t is a standard $\mathbb{P}$-Brownian motion. The short term interest rate r is assumed to be non-random. Then, we have

$$V_t = V_0 \exp\big((r-\kappa-\frac{1}{2}\sigma_V^2)t + \sigma_V B_t\big).$$

We define the default time τ of the firm by

$$\tau := \inf\{t > 0 \mid V_t < L\} \tag{20}$$

where L is a constant liability with $L < V_0$.

Now let

$$Y_t := \log\frac{V_t}{L}. \tag{21}$$

Then, we have easily get the following equations

$$Y_t = \log\frac{V_0}{L} + (r-\kappa-\frac{1}{2}\sigma_V^2)t + \sigma_V B_t, \tag{22}$$

$$\tau = \inf\{t > 0 \mid Y_t < 0\}. \tag{23}$$

By defining the following constants

$$y_0 := \log\frac{V_0}{L}, \quad \nu := r-\kappa-\frac{1}{2}\sigma_V^2, \quad \sigma := \sigma_V, \tag{24}$$

we have

$$Y_t = y_0 + \nu t + \sigma B_t, \tag{25}$$

Figure 2 shows the path of the firm value process V (manager's view), using the same trajectory of Poisson market time shown in Figure 1. It also shows the corresponding market view, i.e. V_{m_t},

5.2 Zero-Coupon Bond

A defaultable zero-coupon bond with zero recovery pays one unit of account at maturity T if no default happened and zero otherwise. Thinking the relation

$$\tau > t \Leftrightarrow \inf_{u\in[0,t]} V_u > L \Leftrightarrow \inf_{u\in[0,t]} Y_u > 0, \tag{26}$$

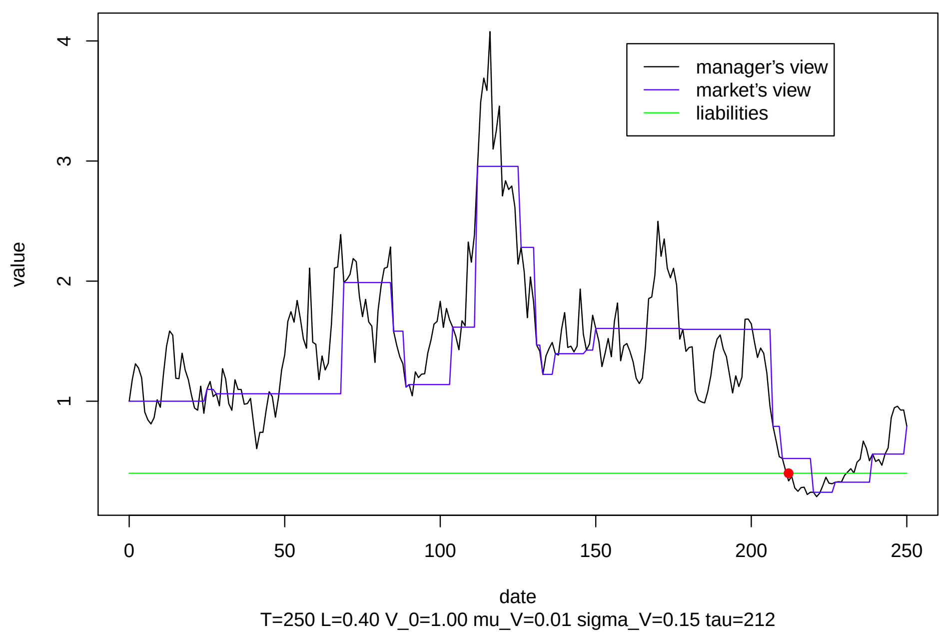

Figure 2: Firm Value Process

the bond price $D(t,T)$ at time $t < T$ is

$$\begin{aligned} D(t,T) &= \mathbb{E}^{\mathbb{P}}\left[e^{-(T-t)r}\mathbb{1}_{\{\tau>T\}} \mid \mathcal{F}_t^m\right] \\ &= e^{-(T-t)r}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\inf_{u\in[0,T]} V_u>L\}} \mid \mathcal{F}_{m_t}\right] \\ &= e^{-(T-t)r}\mathbb{1}_{\{\inf_{u\in[0,m_t]} V_u>L\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\inf_{u\in[m_t,T]} V_u>L\}} \mid \mathcal{F}_{m_t}\right] \\ &= e^{-(T-t)r}\mathbb{1}_{\{\tau>m_t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\inf_{u\in[m_t,T]} Y_u>0\}} \mid \mathcal{F}_{m_t}\right] \\ &= e^{-(T-t)r}\mathbb{1}_{\{\tau>m_t\}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\inf_{u\in[m_t,T]} Y_u>0\}} \mid m_t, Y_{m_t}\right] \\ &= e^{-(T-t)r}\mathbb{1}_{\{\tau>m_t\}}\mathbb{P}\big(\inf_{u\in[m_t,T]} Y_u > 0 \mid m_t, Y_{m_t}\big) \\ &= e^{-(T-t)r}\mathbb{1}_{\{\tau>m_t\}}\left[\Phi\Big(\frac{Y_{m_t}+\nu(T-m_t)}{\sigma\sqrt{T-m_t}}\Big) - e^{-2\nu Y_{m_t}\sigma^{-2}}\Phi\Big(\frac{-Y_{m_t}+\nu(T-m_t)}{\sigma\sqrt{T-m_t}}\Big)\right]. \end{aligned}$$

The last equality comes from Theorem A.8. We also use the insight acquired in Note 4.21, which was proved for a discrete case.

On the other hand, a risk-free bond price that pays one unit of account at maturity T is

$$B(t,T) = e^{-(T-t)r}.$$

Now, the credit spread $S(t,T)$ of the bond is the yield over the risk-free short-rate. Therefore, it becomes on $\{\tau > m_t\}$,

$$\begin{aligned} S(t,T) &= -\frac{1}{T-t}\big(\log D(t,T) - \log B(t,T)\big) \\ &= -\frac{1}{T-t}\left[\Phi\Big(\frac{Y_{m_t}+\nu(T-m_t)}{\sigma\sqrt{T-m_t}}\Big) - e^{-2\nu Y_{m_t}\sigma^{-2}}\Phi\Big(\frac{-Y_{m_t}+\nu(T-m_t)}{\sigma\sqrt{T-m_t}}\Big)\right]. \end{aligned} \tag{27}$$

In order to see a behaviour of the credit spread around the default event, let us see Figure 3. It shows the credit spread $S(t,250)$ calculated by Equation (27). We can see the left-discontinuity at the default point

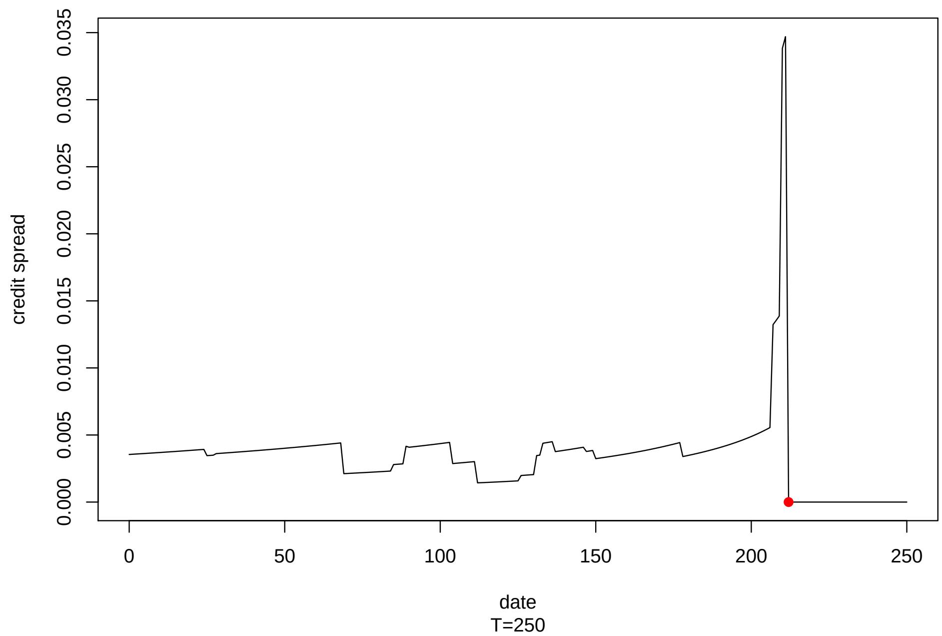

Figure 3: Credit Spread of Defaultable Bond

(the red dot), which is consistent with the empirical observation.

A Appendix

This appendix consists of the known results that are necessary for the discussions in the main text.

A.1 Progressive and Optional Processes

Definition A.1. [Progressive Processes] A process $X = \{X_t\}_{t\in\mathcal{T}}$ is called $\mathbb{F}$-*progressive* if for every $t \in \mathcal{T}$, $X|_{[0,t]_{\mathcal{T}}\times\Omega}$ is $\mathcal{B}[0,t] \otimes \mathcal{F}_t$-measurable. A random set is called $\mathbb{F}$-*progressive* if its indicator function is $\mathbb{F}$-progressive.

Lemma A.2. *[[RW00] Lemma VI.3.3] Every right continuous* $\mathbb{F}$*-adapted process is* $\mathbb{F}$*-progressive.*

Definition A.3. [Optional Processes] The *optional σ-field* with respect to $\mathbb{F}$ is the σ-field $\mathcal{O}^{\mathbb{F}}$ defined on $\mathcal{T} \times \Omega$ such that

$$\mathcal{O}^{\mathbb{F}} := \sigma\{X \mid X = \{X_t\}_{t\in\mathcal{T}} \text{ is an } \mathbb{F}\text{-adapted càdlàg process. }\}. \tag{28}$$

An element of $\mathcal{O}^{\mathbb{F}}$ is called an $\mathbb{F}$-*optional* set. A process $X = \{X_t\}_{t\in\mathcal{T}}$ is called $\mathbb{F}$-*optional* if the map $(t,\omega) \mapsto X_t(\omega)$ is $\mathcal{O}^{\mathbb{F}}$-measurable.

By Lemma A.2, every $\mathbb{F}$-optional process is an $\mathbb{F}$-progressive process, and every $\mathbb{F}$-optional set is an $\mathbb{F}$-progressive set.

The following is one of the standard σ-fields generated by arbitrary random times. You can find it for example in Definition 8.4 in Nikeghbali [Nik06].

Definition A.4. Let τ be a random time. The σ-field $\mathcal{F}_\tau$ is defined by

$$\mathcal{F}_\tau := \sigma\{Z_\tau \mid Z = \{Z_t\}_{t\in\mathcal{T}} \text{ is an } \mathbb{F}\text{-optional process. }\}.$$

The σ-field $\mathcal{F}_\tau$ consists of events which depend on what happens up to and including time τ.

Proposition A.5. *Every random time τ is $\mathcal{F}_\tau$-measurable.*

Proof. Let Z be a process defined by $Z(t,\omega) = t$ for all $t \in \mathcal{T}$ and $\omega \in \Omega$. Then Z is obviously optional and $Z_\tau = \tau$. □

Theorem A.6. *[[DMM92] Théorème XX.27] Let τ_1 and τ_2 be two random times such that $\tau_1 \leq \tau_2$. If τ_1 is $\mathcal{F}_{\tau_2}$-measurable, we have $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$.*

Theorem A.7. *[[RW00] Lemma VI.17.5] If τ is an $\mathbb{F}$-stopping time, then*

$$\mathcal{F}_\tau = \big\{A \in \mathcal{F}_\infty \mid (\forall u \in \mathcal{T}) A \cap \{\tau \leq u\} \in \mathcal{F}_u\big\}. \tag{29}$$

Especially, if there exists a constant $t \in \mathcal{T}$ such that for any $\omega \in \Omega$, $\tau(\omega) = t$, then $\mathcal{F}_\tau = \mathcal{F}_t$.

A.2 Brownian Motion with a Drift

The following theorem is an indispensable tool for calculating default times in markets with complete information.

Theorem A.8. *[[MR05] Corollary B.4.4] Let $\{B_t\}_{t\in\mathbb{R}_+}$ be a standard $\mathbb{F}$-Brownian motion, and $Y = \{Y_t\}_{t\in\mathbb{R}_+}$ be a process satisfying the following equation*

$$Y_t = y_0 + \nu t + \sigma B_t,$$

where $y_0 > 0$, ν and $\sigma > 0$ are given constants. Then for $y \in \mathbb{R}$, we have

$$\mathbb{P}\big(\inf_{s\in[0,t]} Y_s \geq y\big) = \Phi\Big(\frac{-y + y_0 + \nu t}{\sigma\sqrt{t}}\Big) - e^{2\nu(y-y_0)\sigma^{-2}}\Phi\Big(\frac{y - y_0 + \nu t}{\sigma\sqrt{t}}\Big)$$

where the function Φ is the c.d.f. of the standard normal distribution.

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