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A Note on Categorical Risk Measure Theory

Takanori Adachi

Graduate School of International Corporate Strategy,
Hitotsubashi University

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A NOTE ON CATEGORICAL RISK MEASURE THEORY

TAKANORI ADACHI

ABSTRACT. We introduce a category that represents varying risk as well as uncertainty, and give a generalized conditional expectation as a contravariant functor on the category. Then, we reformulate dynamic monetary value measures as a contravariant functor on the category. We show some axioms of dynamic monetary value measures in the classical setting are deduced as theorems in the new formulation, which may be one of the evidences that the *axioms* are natural. We also demonstrate a topology-as-axioms paradigm in order to give a theoretical criteria with which we can pick up appropriate sets of axioms required for monetary value measures to be *good*.

1. INTRODUCTION

The risk measure theory we are formulating is a theory of dynamic (multi-period) monetary risk measures. Since the axiomatization of monetary risk measures was initiated by [ADEH99], many axioms such as law invariance have been presented ([Kus01], [FS11]). Especially after introducing multi-period (or dynamic) versions of monetary risk measures, a lot of investigations have been made so far [ADE$^+$07]. Those investigations are valuable in both theoretical and practical senses. However, it may be expected to have some theoretical criteria of picking appropriate sets of axioms out of them. Thinking about the recent events such as the CDS hedging failure at JP Morgan Chase, the importance of selecting appropriate axioms of monetary risk measures becomes even bigger than before. In this note, we formalize dynamic monetary risk measures in the language of category theory in order to add a new view point to the risk measure theory.

Category theory is an area of study in mathematics that examines in an abstract way the properties of maps (called *morphisms* or *arrows*) satisfying some basic conditions. It has been applied in many

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fields including geometry, logic, computer science and string theory. Even for measure theory, there are some attempts to apply category theory such as [Jac06] or [Bre77]. However, in finance theory, as far as we know, there has been nothing. We will use it for formulating dynamic monetary risk measures and their underlying structure.

In this note, we will stress three points. First is to present a categorical way to handle (dynamic) risk as well as uncertainty, which has the potential to develop several stochastic structures with it. Second is how we can formulate some concepts of dynamic risk measure theory in the structure provided in the first point, and show some *axioms* in the classical setting become *theorems* in our setting. Third is to present a criteria of selecting sets of axioms required for monetary value measure theory in a sheaf-theoretic point of view.

The remainder of this paper consists of four sections.

In Section 2, we provide brief reviews about dynamic risk measure theory and category theory.

In Section 3, we provide a base category with which we handle not just a dynamic (temporal) structure but also uncertainty (spacial) structure in the sense that it handles measure change internally. We define a generalized conditional expectation on the category.

In Section 4, we give a definition of monetary value measures as contravariant functors from the category defined in Section 3 to the category of sets. Then, we will see the resulting monetary value measures satisfy time consistency condition and dynamic programming principle that were introduced as axioms in the old version of dynamic risk measure theory.

In Section 5, we will investigate a possibility of finding an appropriate Grothendieck topology for which monetary value measures satisfying given axioms become sheaves. We also introduce the notion of complete set of axioms with which we give a method to construct a monetary value measure satisfying the axiom from any given monetary value measure.

2. Review of Dynamic Risk Measures and Category Theories

In this section we give a very brief review of dynamic risk measure theory and category theory. Throughout this note, all discussions are under the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.1. **Dynamic Risk Measure Theory.** First, we review the case of one period monetary risk measures.

Definition 2.1. A *one period monetary risk measure* is a function $\rho : L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ satisfying the following axioms

- *Cash invariance:* $(\forall X)(\forall a \in \mathbb{R})\ \rho(X + a) = \rho(X) - a$,
- *Monotonicity:* $(\forall X)(\forall Y)\ X \leq Y \ \Rightarrow\ \rho(X) \geq \rho(Y)$,

- *Normalization:* $\rho(0) = 0$,

where $L^p(\Omega, \mathcal{F}, \mathbb{P})$ is the space of equivalence classes of $\mathbb{R}$-valued random variables which are bounded by the $\|\cdot\|_p$ norm.

Here are examples of one period risk measures.

Example 2.2. [One Period Monetary Risk Measures]

(1) Value at Risk
$$\mathrm{VaR}_\alpha(X) := \inf\{m \in \mathbb{R} \mid \mathbb{P}(X + m < 0) \le \alpha\}$$
(2) Expected shortfall
$$\mathrm{ES}_\alpha(X) := \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(X) du$$

Now, we will define the notion of dynamic monetary risk measures. However, we actually adopt the way of using a *monetary value measure* φ instead of using a *monetary risk measure* ρ below by conforming the manner in recent literature such as [ADE+07] and [KM07], where we have a relation $\varphi(X) = -\rho(X)$ for any possible scenario (i.e. a random variable) X. So from now on, we think a *monetary value measure* φ instead of a monetary risk measure ρ defined by $\varphi(X) := -\rho(X)$.

Definition 2.3. For a σ-field $\mathcal{U} \subset \mathcal{F}$, $L(\mathcal{U}) := L^\infty(\Omega, \mathcal{U}, \mathbb{P}|\mathcal{U})$, is the space of all equivalence classes of bounded $\mathbb{R}$-valued random variables, equipped with the usual sup norm.

Definition 2.4. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be a filtration. A *dynamic monetary value measure* is a collection of functions $\varphi = \{\varphi_t : L(\mathcal{F}_T) \to L(\mathcal{F}_t)\}_{t\in[0,T]}$ satisfying

- *Cash invariance:* $(\forall X \in L(\mathcal{F}_T))(\forall Z \in L(\mathcal{F}_t))\ \varphi_t(X + Z) = \varphi_t(X) + Z$,
- *Monotonicity:* $(\forall X \in L(\mathcal{F}_T))(\forall X \in L(\mathcal{F}_T))\ X \le Y \Rightarrow \varphi_t(X) \le \varphi_t(Y)$,
- *Normalization:* $\varphi_t(0) = 0$.

Note that the directions of some inequalities in Definition 2.1 are different from those of Definition 2.4 because we now monetary value measures instead of monetary risk measures.

Since dynamic monetary value measures treat multi-period situations, we may require some extra axioms to regulate them toward the time dimension. Here are two possible such axioms.

Axiom 2.5. [*Dynamic programming principle*] For $0 \le s \le t \le T$, $(\forall X \in L(\mathcal{F}_T))\ \varphi_s(X) = \varphi_s(\varphi_t(X))$.

Axiom 2.6. [*Time consistency*] For $0 \le s \le t \le T$, $(\forall X, \forall Y \in L(\mathcal{F}_T))\ \varphi_t(X) \le \varphi_t(Y) \ \Rightarrow\ \varphi_s(X) \le \varphi_s(Y)$.

2.2. **Category Theory.** The description about category theory presented in this subsection is very limited. For those who are interested in more detail about category theory, please consult [Mac97].

Definition 2.7. [Categories] A *category* $\mathcal{C}$ consists of a collection $\mathcal{O}_\mathcal{C}$ of *objects* and a collection $\mathcal{M}_\mathcal{C}$ of *arrows* or *morphisms* such that

(1) there are two functions $\mathcal{M}_\mathcal{C} \overset{dom}{\underset{cod}{\rightrightarrows}} \mathcal{O}_\mathcal{C}$.
When $dom(f) = A$ and $cod(f) = B$, we write $f : A \to B$.
We define a so-called *hom-set* of given objects A and B by $\mathrm{Hom}_\mathcal{C}(A, B) := \{f \in \mathcal{M}_\mathcal{C} \mid f : A \to B\}$. We sometimes write $\mathcal{C}(A, B)$ for $\mathrm{Hom}_\mathcal{C}(A, B)$.
(2) for $f : A \to B$ and $g : B \to C$, there is an arrow $g \circ f : A \to C$, called the *composition* of g and f.
(3) every object A is associated with an *identity arrow* $1_A : A \to A$ satisfying $f \circ 1_A = f$ and $1_A \circ g = g$
where $dom(f) = A$ and $cod(g) = A$.

Example 2.8. [Examples of Categories]

(1) **Set** : the category of small sets
- $\mathcal{O}_{\mathbf{Set}}$:= collection of all small sets,
- $\mathcal{M}_{\mathbf{Set}}$:= collection of all functions between small sets.

(2) **Top** : the category of topological spaces
- $\mathcal{O}_{\mathbf{Top}}$:= collection of all topological spaces,
- $\mathcal{M}_{\mathbf{Top}}$:= collection of all continuous functions between topological spaces.

(3) *Opposite category* $\mathcal{C}^{op}$
Let $\mathcal{C}$ be a given category. Then we define its opposite category $\mathcal{C}^{op}$ by the following way.
- $\mathcal{O}_{\mathcal{C}^{op}} := \mathcal{O}_\mathcal{C}$,
- for $A, B \in \mathcal{O}_\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) := \mathrm{Hom}_\mathcal{C}(B, A)$.

Example 2.9. [Preordered Sets as Categories]
A preordered set (sometimes we call it a *proset*) $(S, \leq)$, where the binary relation $\leq$ on S is reflexive and transitive, can be considered as a category defined in the following way.

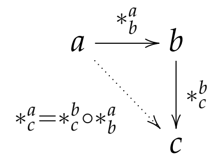
- $\mathcal{O}_S := S$,
- for $a, b \in S$, $\mathrm{Hom}_S(a, b) := \begin{cases} \{*^a_b\} & \text{if } a \leq b, \\ \emptyset & \text{otherwise.} \end{cases}$

We see the correspondence between definitions of prosets and categories below.

(1) Reflexivity vs. identity arrows: $a \leq a$

$$a \xrightarrow{1_a = *^a_a} a$$

(2) Transitivity vs. composition arrows: $a \leq b$ and $b \leq c$ implies $a \leq c$



There are maps between categories, called functors.

Definition 2.10. [Functors] Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of two functions, $F_{\mathcal{O}} : \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{D}}$ and $F_{\mathcal{M}} : \mathcal{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{D}}$ satisfying

(1) $f : A \to B \;\text{ implies }\; F(f) : F(A) \to F(B)$,
(2) $F(g \circ f) = F(g) \circ F(f)$,
(3) $F(1_A) = 1_{F(A)}$.

Definition 2.11. [Contravariant Functors] A functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is called a *contravariant functor*. if two conditions 1 and 2 in Definition 2.10 are replaced by

(1) $f : A \to B \;\text{ implies }\; F(f) : F(B) \to F(A)$,
(2) $F(g \circ f) = F(f) \circ F(g)$.

Here are important examples of contravariant functors, called *representable functors*.

Example 2.12. [Representable Functor]

$$\mathcal{C}^{op} \xrightarrow{Hom_{\mathcal{C}}(-,C)} \mathbf{Set}$$

$$\begin{array}{ccccc} A & & \mathrm{Hom}_{\mathcal{C}}(A, C) & \ni & g \circ f \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(f,C)} & & \uparrow \\ B & & \mathrm{Hom}_{\mathcal{C}}(B, C) & \ni & g \end{array}$$

Now we have maps between functors, called natural transformations.

Definition 2.13. [Natural Transformations] Let $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$ be two functors. A *natural transformation* $\alpha : F \dot{\to} G$ consists of a family of arrows $\langle \alpha_C | C \in \mathcal{O}_{\mathcal{C}} \rangle$ making the following diagram commute:

$$\begin{array}{cccc} C_1 & F(C_1) & \xrightarrow{\alpha_{C_1}} & G(C_1) \\ {\scriptstyle f}\downarrow & {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ C_2 & F(C_2) & \xrightarrow[\alpha_{C_2}]{} & G(C_2) \end{array}$$

Definition 2.14. [Functor Categories] Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor category* $\mathcal{D}^{\mathcal{C}}$ is the category such that

- $\mathcal{O}_{\mathcal{D}^{\mathcal{C}}} :=$ collection of all functors from $\mathcal{C}$ to $\mathcal{D}$,
- $\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G) :=$ collection of all natural transformations from F to G.

3. Generalized Conditional Expectations

We fix a measurable space $(\Omega, \mathcal{F})$ in the rest of this note.

In this section, we give a category called χ which will be a base category throughout this note, and define a generalized conditional expectation functor on it.

Definition 3.1. [Category χ] Let $\chi := \chi(\Omega, \mathcal{F})$ be the set of all pairs of the form $(\mathcal{G}, \mathbb{P})$ where $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$ and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. For an element $\mathcal{U} \in \chi$, we denote its σ-field and probability measure by $\mathcal{F}_{\mathcal{U}}$ and $\mathbb{P}_{\mathcal{U}}$, respectively. That is, $\mathcal{U} = (\mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{U}})$.

Let us introduce a binary relation $\leq_\chi$ on χ by for $\mathcal{U}$ and $\mathcal{V}$ in χ,

$$\mathcal{V} \leq_\chi \mathcal{U} \quad \text{iff} \quad \mathcal{F}_{\mathcal{V}} \subset \mathcal{F}_{\mathcal{U}} \text{ and } \mathbb{P}_{\mathcal{V}} \gg \mathbb{P}_{\mathcal{U}} \tag{3.1}$$

where $\mathbb{P}_{\mathcal{V}} \gg \mathbb{P}_{\mathcal{U}}$ means that $\mathbb{P}_{\mathcal{U}}$ is absolutely continuous to $\mathbb{P}_{\mathcal{V}}$. Then, obviously the system $(\chi, \leq_\chi)$ is a preordered set. Hence by Example 2.9 we can think χ as a category.

We simply denote the unique arrow $*_{\mathcal{U}}^{\mathcal{V}}$ between objects $\mathcal{V}$ and $\mathcal{U}$ of χ with $\mathcal{V} \leq_\chi \mathcal{U}$ by $*$ unless there is a risk of ambiguity.

We may be able to think the category χ having two dimensions; one is a time dimension or risk dimension that is represented in a horizontal direction in Diagram 3.1, and the other is a space dimension or uncertainty dimension representing in a vertical direction.

Note that for objects $\mathcal{U}, \mathcal{V} \in \chi$, $\mathcal{U}$ is isomorphic to $\mathcal{V}$ (we write this by $\mathcal{U} \simeq \mathcal{V}$) if and only if $\mathcal{F}_{\mathcal{V}} = \mathcal{F}_{\mathcal{U}}$ and $\mathbb{P}_{\mathcal{V}} \approx \mathbb{P}_{\mathcal{U}}$ (equivalent) .

Now let $\mathcal{V} \to \mathcal{U}$ be an arrow in χ. Then by definition, we have $\mathbb{P}_{\mathcal{U}} \ll \mathbb{P}_{\mathcal{V}}$, in other words, $\mathbb{P}_{\mathcal{U}}(A) = 0$ whenever $\mathbb{P}_{\mathcal{V}}(A) = 0$ for $A \in \mathcal{F}$. Therefore, for any $\mathcal{F}$-measurable function X on Ω, we have $[X]_{\mathbb{P}_{\mathcal{V}}} \subset [X]_{\mathbb{P}_{\mathcal{U}}}$. This fact makes the following definition be well-defined.

Definition 3.2. [Functor L]

A functor $L : \chi \to \mathbf{Set}$ is defined by for $\mathcal{V} \to \mathcal{U}$ in χ,

$$\begin{array}{ccccccc}
\mathcal{V} & \stackrel{L}{\longmapsto} & L_{\mathcal{V}} & := & L^\infty(\Omega, \mathcal{F}_{\mathcal{V}}, \mathbb{P}_{\mathcal{V}}|_{\mathcal{F}_{\mathcal{V}}}) & \ni & [X]_{\mathbb{P}_{\mathcal{V}}|_{\mathcal{F}_{\mathcal{V}}}} \\
\downarrow & & \downarrow{\scriptstyle L_{\mathcal{U}}^{\mathcal{V}}} & & & & \downarrow{\scriptstyle L_{\mathcal{U}}^{\mathcal{V}}} \\
\mathcal{U} & \stackrel{L}{\longmapsto} & L_{\mathcal{U}} & := & L^\infty(\Omega, \mathcal{F}_{\mathcal{U}}, \mathbb{P}_{\mathcal{U}}|_{\mathcal{F}_{\mathcal{U}}}) & \ni & [X]_{\mathbb{P}_{\mathcal{U}}|_{\mathcal{F}_{\mathcal{U}}}}
\end{array}$$

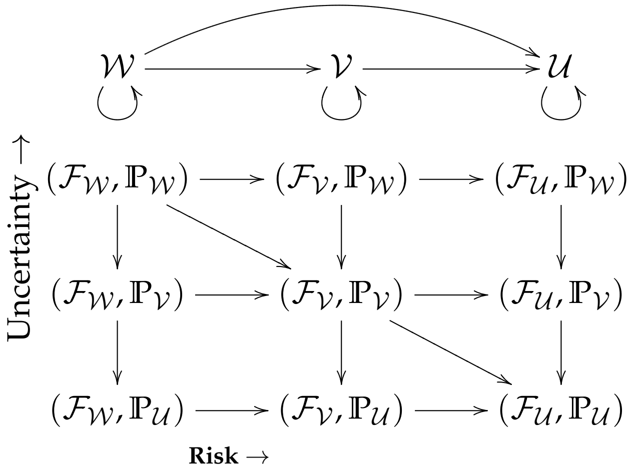


DIAGRAM 3.1

where $L^\infty(\Omega, \mathcal{F}_\mathcal{U}, \mathbb{P}_\mathcal{U}|_{\mathcal{F}_\mathcal{U}})$ is the Banach space defined as a quotient space of all $\mathcal{F}_\mathcal{U}$-measurable functions on Ω under the equivalent relation $\sim_{\mathbb{P}_\mathcal{U}}$ defined by $X \sim_{\mathbb{P}_\mathcal{U}} Y$ iff $X = Y$ $\mathbb{P}_\mathcal{U}$-a.s..

Proposition 3.3. *For* $\mathcal{W} \to \mathcal{V} \to \mathcal{U}$ *in* χ *and* $X \in L_\mathcal{U}$*,*

(1) $\mathbb{E}^{\mathbb{P}_\mathcal{U}}[X|\mathcal{F}_\mathcal{V}]\frac{d\mathbb{P}_\mathcal{U}}{d\mathbb{P}_\mathcal{V}}|_{\mathcal{F}_\mathcal{V}} = \mathbb{E}^{\mathbb{P}_\mathcal{V}}[X\frac{d\mathbb{P}_\mathcal{U}}{d\mathbb{P}_\mathcal{V}}|_{\mathcal{F}_\mathcal{U}}|\mathcal{F}_\mathcal{V}]$ $\quad \mathbb{P}_\mathcal{V}$*-a.s.,*

(2) $\frac{d\mathbb{P}_\mathcal{V}}{d\mathbb{P}_\mathcal{W}}|_{\mathcal{F}_\mathcal{U}} \times \frac{d\mathbb{P}_\mathcal{U}}{d\mathbb{P}_\mathcal{V}}|_{\mathcal{F}_\mathcal{U}} = \frac{d\mathbb{P}_\mathcal{U}}{d\mathbb{P}_\mathcal{W}}|_{\mathcal{F}_\mathcal{U}}$ $\quad \mathbb{P}_\mathcal{U}$*-a.s.*

where $\frac{d\mathbb{P}_\mathcal{U}}{d\mathbb{P}_\mathcal{V}}$ *is a Radon-Nikodym derivative of* $\mathbb{P}_\mathcal{U}$ *with respect to* $\mathbb{P}_\mathcal{V}$*.*

Proof. (1) When $\mathbb{Q} \ll \mathbb{P}$ and $\mathcal{G} \subset \mathcal{F}$, we have

$$\frac{d\mathbb{Q}}{d\mathbb{P}} |_\mathcal{G} = \mathbb{E}^P[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{G}] \quad \mathbb{P}\text{-a.s.} \tag{3.2}$$

and

$$\mathbb{E}^\mathbb{Q}[X \mid \mathcal{G}] = \frac{\mathbb{E}^P[X\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{G}]}{\mathbb{E}^P[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{G}]} \quad \mathbb{Q}\text{-a.s.} \tag{3.3}$$

by Proposition A.11 and Proposition A.12 in [FS11]. Then, by (3.2) and since X is $\mathcal{F}_\mathcal{U}$-measurable, we have with $\mathbb{P}_\mathcal{V}$-a.s.,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}_\mathcal{V}}[X\frac{d\mathbb{P}_\mathcal{U}}{d\mathbb{P}_\mathcal{V}} |_{\mathcal{F}_\mathcal{U}}| \mathcal{F}_\mathcal{V}] &= \mathbb{E}^{\mathbb{P}_\mathcal{V}}[X\mathbb{E}^{\mathbb{P}_\mathcal{V}}[\frac{d\mathbb{P}_\mathcal{U}}{d\mathbb{P}_\mathcal{V}} \mid \mathcal{F}_\mathcal{U}] \mid \mathcal{F}_\mathcal{V}] \\ &= \mathbb{E}^{\mathbb{P}_\mathcal{V}}[\mathbb{E}^{\mathbb{P}_\mathcal{V}}[X\frac{d\mathbb{P}_\mathcal{U}}{d\mathbb{P}_\mathcal{V}} \mid \mathcal{F}_\mathcal{U}] \mid \mathcal{F}_\mathcal{V}] \\ &= \mathbb{E}^{\mathbb{P}_\mathcal{V}}[X\frac{d\mathbb{P}_\mathcal{U}}{d\mathbb{P}_\mathcal{V}} \mid \mathcal{F}_\mathcal{V}].\end{aligned}$$

Therefore, again by (3.2) and (3.3), we get the desired equation.

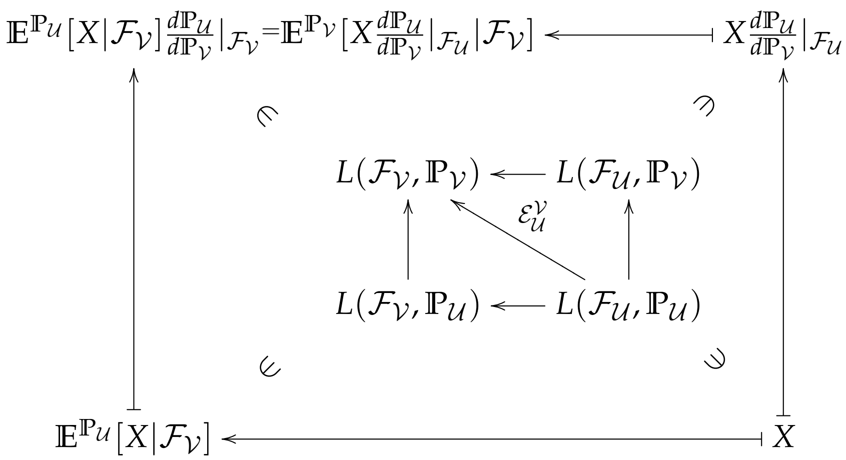


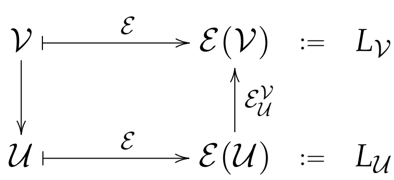
DIAGRAM 3.2

(2) By (3.2), (3.3) and again by (3.2), we have with $\mathbb{P}_{\mathcal{U}}$-a.s.,

$$\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{U}}} = \mathbb{E}^{\mathbb{P}_{\mathcal{V}}}[\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}} \mid \mathcal{F}_{\mathcal{U}}] = \frac{\mathbb{E}^{\mathbb{P}_{\mathcal{W}}}[\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}\frac{d\mathbb{P}_{\mathcal{V}}}{d\mathbb{P}_{\mathcal{W}}} \mid \mathcal{F}_{\mathcal{U}}]}{\mathbb{E}^{\mathbb{P}_{\mathcal{W}}}[\frac{d\mathbb{P}_{\mathcal{V}}}{d\mathbb{P}_{\mathcal{W}}} \mid \mathcal{F}_{\mathcal{U}}]}$$
$$= \frac{\mathbb{E}^{\mathbb{P}_{\mathcal{W}}}[\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{W}}} \mid \mathcal{F}_{\mathcal{U}}]}{\mathbb{E}^{\mathbb{P}_{\mathcal{W}}}[\frac{d\mathbb{P}_{\mathcal{V}}}{d\mathbb{P}_{\mathcal{W}}} \mid \mathcal{F}_{\mathcal{U}}]} = \frac{\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{W}}}|_{\mathcal{F}_{\mathcal{U}}}}{\frac{d\mathbb{P}_{\mathcal{V}}}{d\mathbb{P}_{\mathcal{W}}}|_{\mathcal{F}_{\mathcal{U}}}}.$$

□

Definition 3.4. [Generalized Conditional Expectation] A *generalized conditional expectation* is a contravariant functor $\mathcal{E} : \chi^{op} \to$ **Set** defined by for $\mathcal{V} \to \mathcal{U}$ in χ,



where

$$\mathcal{E}^{\mathcal{V}}_{\mathcal{U}}(X) := \mathbb{E}^{\mathbb{P}_{\mathcal{U}}}[X|\mathcal{F}_{\mathcal{V}}]\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{V}}} \tag{3.4}$$

for $X \in L_{\mathcal{U}}$.

Note that $\mathcal{E}^{\mathcal{V}}_{\mathcal{U}}$ in Definition 3.4 is well-defined by Proposition 3.3. See also Diagram 3.2 and Diagram 3.3.

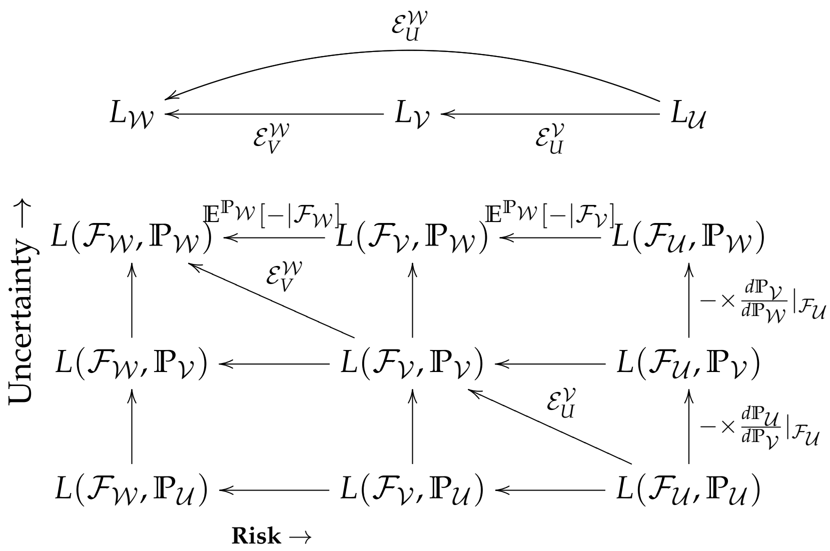


DIAGRAM 3.3

4. MONETARY VALUE MEASURES

Definition 4.1. [Monetary Value Measures] A *monetary value measure* is a contravariant functor

$$\varphi : \chi^{op} \to \mathbf{Set}$$

satisfying the following two conditions:

(1) for $\mathcal{U} \in \chi$, $\varphi(\mathcal{U}) := L_{\mathcal{U}}$,
(2) for $\mathcal{V} \to \mathcal{U}$ in χ, the map $\varphi^{\mathcal{V}}_{\mathcal{U}} := \varphi(\mathcal{V} \to \mathcal{U}) : L_{\mathcal{U}} \to L_{\mathcal{V}}$ satisfies
   - *Cash invariance:* $(\forall X \in L_{\mathcal{U}})(\forall Z \in L_{\mathcal{V}})\ \varphi^{\mathcal{V}}_{\mathcal{U}}(X + L^{\mathcal{V}}_{\mathcal{U}}(Z)) = \varphi^{\mathcal{V}}_{\mathcal{U}}(X) + Z$ $\mathbb{P}_{\mathcal{V}}$-a.s.,
   - *Monotonicity:* $(\forall X \in L_{\mathcal{U}})(\forall Y \in L_{\mathcal{U}})\ X \leq Y \Rightarrow \varphi^{\mathcal{V}}_{\mathcal{U}}(X) \leq \varphi^{\mathcal{V}}_{\mathcal{U}}(Y)$ $\mathbb{P}_{\mathcal{V}}$-a.s.,
   - *Normalization:* $\varphi^{\mathcal{V}}_{\mathcal{U}}(0_{L_{\mathcal{U}}}) = 0_{L_{\mathcal{V}}}$ $\mathbb{P}_{\mathcal{V}}$-a.s. if $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$.

At this point, we do not require the monetary value measures to satisfy some familiar conditions such as concavity, positive homogeneity or law invariance. Instead of doing so, we want to see what kind of properties are deduced from this minimal setting.

The most crucial key points of Definition 4.1 is that φ does not move only toward time direction but also moves over several absolutely continuous probability measures *internally*. This means we have a possibility to develop risk measures including uncertainty within this formulation.

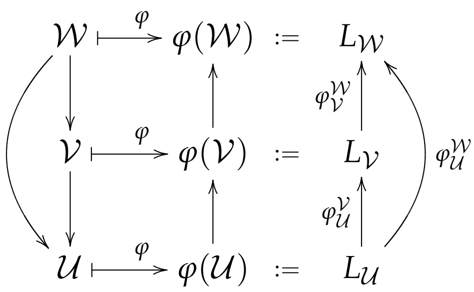


DIAGRAM 4.1

Another key points of Definition 4.1 is that φ is a contravariant functor. So, for any triple $\mathcal{W} \to \mathcal{V} \to \mathcal{U}$ in χ, we have, as seeing in Diagram 4.1,

$$\varphi^{\mathcal{U}}_{\mathcal{U}} = 1_{L_{\mathcal{U}}} \quad \text{and} \quad \varphi^{\mathcal{W}}_{\mathcal{V}} \circ \varphi^{\mathcal{V}}_{\mathcal{U}} = \varphi^{\mathcal{W}}_{\mathcal{U}}. \tag{4.1}$$

Example 4.2. [Entropic Value Measure] For a non-zero real number λ, an *entropic value measure* is a contravariant functor $\varphi : \chi^{op} \to \mathbf{Set}$ defined by

$$\varphi(\mathcal{U}) := L_{\mathcal{U}} \quad \text{and} \quad \varphi^{\mathcal{V}}_{\mathcal{U}}(X) := \lambda^{-1} \log \mathcal{E}^{\mathcal{V}}_{\mathcal{U}}(e^{\lambda X})$$

for $\mathcal{V} \to \mathcal{U}$ in χ and $X \in L_{\mathcal{U}}$. Then, it is easy to see that the contravariant functor φ is well-defined and is a monetary value measure.

Now in case $\mathcal{F}_{\mathcal{V}} = \mathcal{F}_{\mathcal{U}}$, we have

$$\begin{aligned}
\varphi^{\mathcal{V}}_{\mathcal{U}}(X) &= \lambda^{-1} \log \mathcal{E}^{\mathcal{V}}_{\mathcal{U}}(e^{\lambda X}) \\
&= \lambda^{-1} \log \big(e^{\lambda X} \frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{V}}}\big) \\
&= X + \lambda^{-1} \log \big(\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{V}}}\big).
\end{aligned}$$

Especially, we have $\varphi^{\mathcal{V}}_{\mathcal{U}}(0_{L_{\mathcal{U}}}) = \lambda^{-1} \log \big(\frac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}_{\mathcal{V}}}|_{\mathcal{F}_{\mathcal{V}}}\big)$, which is not $0_{L_{\mathcal{V}}}$ unless $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$ on $\mathcal{F}_{\mathcal{V}}$.

This is the reason we require the assumption $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$ in the normalization condition in Definition 4.1.

Here are some properties of monetary value measures.

Proposition 4.3. *Let $\varphi : \chi^{op} \to \mathbf{Set}$ be a monetary value measure, and $\mathcal{W} \to \mathcal{V} \to \mathcal{U}$ be arrows in χ.*

(1) If $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$, we have $\varphi^{\mathcal{V}}_{\mathcal{U}} \circ L^{\mathcal{V}}_{\mathcal{U}} = 1_{L_{\mathcal{V}}}$.
(2) Idempotentness: If $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$, we have $\varphi^{\mathcal{V}}_{\mathcal{U}} \circ L^{\mathcal{V}}_{\mathcal{U}} \circ \varphi^{\mathcal{V}}_{\mathcal{U}} = \varphi^{\mathcal{V}}_{\mathcal{U}}$.
(3) Local property: $(\forall X \in L_{\mathcal{U}})(\forall Y \in L_{\mathcal{U}})(\forall A \in \mathcal{V})$ $\varphi^{\mathcal{V}}_{\mathcal{U}}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) = \mathbb{1}_A \varphi^{\mathcal{V}}_{\mathcal{U}}(X) + \mathbb{1}_{A^c} \varphi^{\mathcal{V}}_{\mathcal{U}}(Y)$.

(4) *Dynamic programming principle:* If $\mathbb{P}_{\mathcal{V}} = \mathbb{P}_{\mathcal{U}}$, we have $\varphi_{\mathcal{U}}^{\mathcal{W}} = \varphi_{\mathcal{U}}^{\mathcal{W}} \circ L_{\mathcal{U}}^{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}}$.

(5) *Time consistency:* $(\forall X \in L_{\mathcal{U}})(\forall Y \in L_{\mathcal{U}})\ \varphi_{\mathcal{U}}^{\mathcal{V}}(X) \leq \varphi_{\mathcal{U}}^{\mathcal{V}}(Y) \Rightarrow \varphi_{\mathcal{U}}^{\mathcal{W}}(X) \leq \varphi_{\mathcal{U}}^{\mathcal{W}}(Y)$.

Proof. (1) For $X \in L_{\mathcal{V}}$, we have by cash invariance and normalization, $\varphi_{\mathcal{U}}^{\mathcal{V}}(L_{\mathcal{U}}^{\mathcal{V}}(X)) = \varphi_{\mathcal{U}}^{\mathcal{V}}(0_{L_{\mathcal{U}}} + L_{\mathcal{U}}^{\mathcal{V}}(X)) = \varphi_{\mathcal{U}}^{\mathcal{V}}(0_{L_{\mathcal{U}}}) + X = X$.

(2) Immediate by (1).

(3) First, we show that for any $A \in \mathcal{V}$,

$$\mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(X) = \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X). \tag{4.2}$$

Since $X \in L^{\infty}(\Omega, \mathcal{U}, \mathbb{P})$, we have $|X| \leq \|X\|_{\infty}$. Therefore,

$$\mathbb{1}_A X - \mathbb{1}_{A^c}\|X\|_{\infty} \leq \mathbb{1}_A X + \mathbb{1}_{A^c} X \leq \mathbb{1}_A X + \mathbb{1}_{A^c}\|X\|_{\infty}.$$

Then, by cash invariance and monotonicity,

$$\begin{aligned}
\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) - \mathbb{1}_{A^c}\|X\|_{\infty} &= \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X - \mathbb{1}_{A^c}\|X\|_{\infty}) \\
&\leq \varphi_{\mathcal{U}}^{\mathcal{V}}(X) \\
&\leq \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X + \mathbb{1}_{A^c}\|X\|_{\infty}) = \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) + \mathbb{1}_{A^c}\|X\|_{\infty}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) &= \mathbb{1}_A(\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) - \mathbb{1}_{A^c}\|X\|_{\infty}) \\
&\leq \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(X) \\
&\leq \mathbb{1}_A(\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) + \mathbb{1}_{A^c}\|X\|_{\infty}) = \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X).
\end{aligned}$$

Therefore, we get (4.2).

Next by using (4.2) twice, we have

$$\begin{aligned}
\varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) &= \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) + \mathbb{1}_{A^c} \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y) \\
&= \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A(\mathbb{1}_A X + \mathbb{1}_{A^c} Y)) + \mathbb{1}_{A^c} \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_{A^c}(\mathbb{1}_A X + \mathbb{1}_{A^c} Y)) \\
&= \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_A X) + \mathbb{1}_{A^c} \varphi_{\mathcal{U}}^{\mathcal{V}}(\mathbb{1}_{A^c} Y) \\
&= \mathbb{1}_A \varphi_{\mathcal{U}}^{\mathcal{V}}(X) + \mathbb{1}_{A^c} \varphi_{\mathcal{U}}^{\mathcal{V}}(Y).
\end{aligned}$$

(4) By (2) and (4.1), we have

$$\begin{aligned}
\varphi_{\mathcal{U}}^{\mathcal{W}} &= \varphi_{\mathcal{V}}^{\mathcal{W}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}} = \varphi_{\mathcal{V}}^{\mathcal{W}} \circ (\varphi_{\mathcal{U}}^{\mathcal{V}} \circ L_{\mathcal{U}}^{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}}) \\
&= (\varphi_{\mathcal{V}}^{\mathcal{W}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}}) \circ (L_{\mathcal{U}}^{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}}) = \varphi_{\mathcal{U}}^{\mathcal{W}} \circ L_{\mathcal{U}}^{\mathcal{V}} \circ \varphi_{\mathcal{U}}^{\mathcal{V}}.
\end{aligned}$$

(5) Assume $\varphi_{\mathcal{U}}^{\mathcal{V}}(X) \leq \varphi_{\mathcal{U}}^{\mathcal{V}}(Y)$. Then, by monotonicity and (4.1),

$$\varphi_{\mathcal{U}}^{\mathcal{W}}(X) = \varphi_{\mathcal{V}}^{\mathcal{W}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(X)) \leq \varphi_{\mathcal{V}}^{\mathcal{W}}(\varphi_{\mathcal{U}}^{\mathcal{V}}(Y)) = \varphi_{\mathcal{U}}^{\mathcal{W}}(Y).$$

□

In Proposition 4.3, two properties, dynamic programming principle and time consistency are usually introduced as axioms ([DS06]). But, we derive them naturally here from the fact that the monetary value measure is a contravariant functor as a proposition. This may be seen as an evidence that the two *axioms* are natural.

Before ending this section, we mention an interpretation of one of the most important theorems in category theory called the Yoneda lemma in our setting.

Theorem 4.4. *[The Yoneda Lemma] For any monetary value measure* $\varphi : \chi^{op} \to \mathbf{Set}$ *and an object* $\mathcal{U}$ *in* χ*, there exists a bijective correspondence* $y_{\varphi,\mathcal{U}}$ *specified by the following diagram:*

$$y_{\varphi,\mathcal{U}} : \mathrm{Nat}(\mathrm{Hom}_\chi(-,\mathcal{U}),\varphi) \xrightarrow{\cong} L_{\mathcal{U}}$$

$$\alpha \longmapsto \alpha_{\mathcal{U}}(*^{\mathcal{U}}_{\mathcal{U}})$$

$$\tilde{X} \longleftarrow\!\!\shortmid X$$

where $\tilde{X}$ *is a natural transformation defined by for any* $\mathcal{V} \to \mathcal{U}$ *in* χ*,* $\tilde{X}_{\mathcal{V}}(*^{\mathcal{V}}_{\mathcal{U}}) := \varphi^{\mathcal{V}}_{\mathcal{U}}(X)$*. Moreover, the correspondence is natural in both* φ *and* $\mathcal{U}$*.*

Let us see the representable functor $\mathrm{Hom}_\chi(-,\mathcal{U})$ as a generalized *time domain* with the time horizon $\mathcal{U}$. Then a natural transformation from $\mathrm{Hom}_\chi(-,\mathcal{U})$ to φ can be seen as a *stochastic process* that is (in a sense) adapted to φ, and its corresponding $\mathcal{F}_{\mathcal{U}}$-measurable random variable represents a terminal value (payoff) at the horizon.

The Yoneda lemma says that we have a bijective correspondence between those stochastic processes and random variables.

5. Monetary Value Measures as Sheaves

In general, a contravariant functor $\rho : \mathcal{C}^{op} \to \mathbf{Set}$ is called a *presheaf* for a category $\mathcal{C}$. By definition, a monetary value measure is a presheaf for χ. The name *presheaf* suggests that it is related to another concept *sheaves*, which is a quite important concept in some classical branches in mathematics such as algebraic topology. [MM92]. So, what makes a presheaf be a sheaf?

For a given set, a topology defined on it provides a criteria to identify good (= continuous) functions within functions on the set. In a similar way, there is a concept called a *Grothendieck topology* defined on a given category that gives a criteria to identify good presheaves (= sheaves) among presheaves on the category. In both cases, a (Grothendieck) topology can be seen as a vehicle to identify good functions (presheaves) among general functions (presheaves).

set	category	χ
topology	Grothendieck topology	
function	presheaf	value measure
continuous function	sheaf	value measure *satisfying axioms*
weakest topology	largest Grothendieck topology	

TABLE 5.1. topology-as-axioms paradigm

On the other hand, if we have a set of functions that we want to make *good* (= continuous), we can find the weakest topology that makes the functions continuous. In a similar way, if we have a set of presheaves that we want to make *good*, it is known that we can pick a Grothendieck topology with which the presheaves become sheaves. See Table 5.1 for the analogy.

Since a monetary value measure is a presheaf, if we have a set of *good* monetary value measures (= the monetary value measures that satisfy a given set of axioms), we may find a Grothendieck topology with which the monetary value measures become sheaves. We will see a concrete shape of the Grothendieck topology in Section 5.1.

Now suppose we have a weak topology that makes given functions continuous. This, however, does not imply the fact that any continuous function w.r.t. the topology is contained in the originally given functions. Similarly, Suppose that we have a Grothendieck topology that makes all monetary value measures satisfying a given set of axioms sheaves. It, however, does not mean that any sheaf w.r.t. the Grothendieck topology satisfies the given set of axioms. We will investigate this situation in Section 5.2.

5.1. **A Grothendieck Topology as Axioms.** The following two subsections are devoted to standard or straightforward discussions in the context of sheaf theory. However, we think it is worth to record those stuff here since they are new in the context of risk measure theory.

In this subsection, we see a concrete shape of the Grothendieck topology with which all monetary value measures satisfying a given set of axioms become sheaves.

First, we review two concepts of Grothendieck typologies and sheaves.

Definition 5.1. [Grothendieck Topology] A *Grothendieck topology J* on χ consists in giving, for each object $\mathcal{U} \in \chi$, a family $J(\mathcal{U})$ of subfunctors of the representable functor $\mathrm{Hom}_{\chi}(-,\mathcal{U})$, satisfying the following axioms:

(1) for every $\mathcal{U} \in \chi$, $\mathrm{Hom}_{\chi}(-,\mathcal{U}) \in J(\mathcal{U})$.

$$\begin{array}{ccccc} R^{\mathcal{V}} & \rightarrowtail & \mathrm{Hom}_{\chi}(-,\mathcal{V}) & & \mathcal{V} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\chi}(-,*)} & & \downarrow{\scriptstyle *} \\ R & \rightarrowtail & \mathrm{Hom}_{\chi}(-,\mathcal{U}) & & \mathcal{U} \end{array}$$

DIAGRAM 5.1

(2) for any $\mathcal{V} \xrightarrow{*} \mathcal{U}$ in χ and $R \in J(\mathcal{U})$, the presheaf $R^{\mathcal{V}}$ defined as a pullback in Diagram 5.1 belongs to $J(\mathcal{V})$.

(3) let $R \rightarrowtail \mathrm{Hom}_{\chi}(-,\mathcal{U})$ and $Q \in J(\mathcal{U})$. If we have $R^{\mathcal{V}} \in J(\mathcal{V})$ for the pullback defined in Diagram 5.1 whenever $\mathcal{V} \xrightarrow{*} \mathcal{U}$ is in $Q(\mathcal{V})$, then $R \in J(\mathcal{U})$.

Since Diagram 5.1 is a pullback in **Set** and the cardinality of the set $\mathrm{Hom}_{\chi}(\mathcal{V},\mathcal{U})$ is at most 1, we have for every $\mathcal{W} \to \mathcal{V} \in \chi$,

$$R^{\mathcal{V}}(\mathcal{W}) = \begin{cases} \{*^{\mathcal{W}}_{\mathcal{V}}\} & \text{if } R(\mathcal{W}) = \{*^{\mathcal{W}}_{\mathcal{U}}\}, \\ \emptyset & \text{if } R(\mathcal{W}) = \emptyset. \end{cases} \tag{5.1}$$

Here is a well-known property of Grothendieck topologies.

Theorem 5.2. *Let $\{J_a \mid a \in A\}$ be a collection of Grothendieck topologies on χ. Then a system J defined by $J(\mathcal{U}) := \bigcap_{a \in A} J_a(\mathcal{U})$ for every object $\mathcal{U} \in \chi$ is a Grothendieck topology. We write this J by $\bigcap_{a \in A} J_a$.*

Now, we can introduce the concept of sheaves.

Definition 5.3. [Sheaves] A presheaf φ on χ is called a *sheaf* for a Grothendieck topology J when, given $\mathcal{U} \in \chi$ and $R \in J(\mathcal{U})$, every natural transformation $X : R \dot{\to} \varphi$ extends uniquely to $\mathrm{Hom}_{\chi}(-,\mathcal{U})$.

In the rest of this subsection, we will try to find a Grothendieck topology for which a given class of monetary value measures specified by a given set of (extra) axioms are sheaves.

The following proposition assures the existence of a Grothendieck topology making a given monetary value measure a sheaf.

Proposition 5.4. *Let $\varphi \in \mathbf{Set}^{\chi^{op}}$ be a monetary value measure and $\mathcal{U} \in \chi$. Define a set of subfunctors $J_{\varphi}(\mathcal{U})$ by*

(5.2)

$$J_{\varphi}(\mathcal{U}) := \left\{ R \rightarrowtail \mathrm{Hom}_{\chi}(-,\mathcal{U}) \;\middle|\; ({}^{\forall}\mathcal{V} \xrightarrow{*} \mathcal{U} \text{ in } \chi)\ \begin{array}{ccc} R^{\mathcal{V}} & \rightarrowtail & \mathrm{Hom}_{\chi}(-,\mathcal{V}) \\ {\scriptstyle \forall X}\downarrow & \swarrow{\scriptstyle \exists ! Y} & \\ \varphi & & \end{array} \right\}$$

where $R^{\mathcal{V}}$ is a presheaf making Diagram 5.1 a pullback. Then, J_{φ} is the largest Grothendieck topology for which φ is a sheaf.

Proof. Refer Example 3.2.14c in [Bor94]. □

By combining Proposition 5.4 and Theorem 5.2, we have the following corollary.

Corollary 5.5. *Let $\mathcal{M} \subset \mathbf{Set}^{\chi^{op}}$ be the collection of all monetary value measures satisfying a given set of axioms. Then, there exists a Grothendieck topology for which all monetary value measures in $\mathcal{M}$ are sheaves, where the topology is largest among topologies representing the axioms. We write the topology by $J_{\mathcal{M}}$.*

Proof. Let $J_{\mathcal{M}} := \bigcap_{\varphi \in \mathcal{M}} J_\varphi$. Then, it is the largest Grothendieck topology for which every monetary value measure in $\mathcal{M}$ is a sheaf. □

5.2. **Complete sets of Axioms.** Let $\mathcal{A}$ be a *fixed* set of axioms. Then, for a given arbitrary monetary value measure φ, can we make a *good* alternative for it? In other words, can we find a monetary value measure that satisfies $\mathcal{A}$ and is the best approximation of the original φ? This is the theme of this subsection.

For a Grothendieck topology J on χ, define $Sh(\chi, J) \subset \mathrm{Set}^{\chi^{op}}$ to be a full subcategory whose objects are all sheaves for J. Then, it is well-known that there exists a *left adjoint* π_J in the following diagram.

$$
\begin{array}{ccc}
Sh(\chi, J) & \underset{\pi_J}{\rightleftarrows} & \mathrm{Set}^{\chi^{op}} \\
\Cup & & \Cup \\
\pi_J(\varphi) & \longleftarrow\!\!\shortmid & \varphi
\end{array}
\tag{5.3}
$$

The functor π_J is well-known with the name *sheafification* functor, which comes with the following limit cone:

(5.4)

$$
\begin{array}{ccccc}
\dots \longrightarrow \mathrm{Nat}(R, \varphi) & & \xrightarrow{\quad \mathrm{Nat}(\alpha,\varphi) \quad} & & \mathrm{Nat}(Q, \varphi) \longrightarrow \dots \\
& \searrow^{S_R^\varphi} & & \swarrow_{S_Q^\varphi} & \\
\pi_J(\varphi)(\mathcal{U}) & := & \mathrm{colim}_{R \in J(\mathcal{U})} \mathrm{Nat}(R, \varphi) & &
\end{array}
$$

for $\alpha : Q \to R$ in $\mathrm{Set}^{\chi^{op}}$. It also satisfies the following theorem.

Theorem 5.6. *(1) $\pi_J(\varphi)$ is a sheaf for J.*
(2) If φ is a sheaf for J, then $\pi_J(\varphi) \simeq \varphi$.

Theorem 5.6 suggests that for an arbitrary monetary value measure, the sheafification functor provides one of its closest monetary value measures that *may* satisfy the given set of axioms. To make this certain, we need a following definition.

Definition 5.7. Let $\mathcal{A}$ be a set of axioms for monetary value measures.

(1) $\mathcal{M}(\mathcal{A}) :=$ the collection of all monetary value measures satisfying $\mathcal{A}$.
(2) $\mathcal{M}_0 :=$ the collection of all monetary value measures.
(3) $\mathcal{A}$ is called *complete* if

$$\pi_{J_{\mathcal{M}(\mathcal{A})}}(\mathcal{M}_0) \subset \mathcal{M}(\mathcal{A}). \tag{5.5}$$

By Theorem 5.6, we have the following main result.

Theorem 5.8. *Let $\mathcal{A}$ be a complete set of axioms. Then, for a monetary value measure $\varphi \in \mathcal{M}_0$, $\pi_{J_{\mathcal{M}(\mathcal{A})}(\varphi)}$ is the monetary value measure that is the best approximation satisfying axioms $\mathcal{A}$.*

Now, we want to expect that some of the well-known sets of axioms such as those for concave monetary value measures are complete. If we restrict the category χ to the category that is not allowed to vary its probability measures, i.e. no uncertainty version, then we have a counterexample for a quite small Ω [Ada12]. However, we have no significant result so far for the current version of χ that accepts uncertainty.

6. Conclusion

We introduced a category χ that represents varying risk as well as uncertainty. We gave a generalized conditional expectation as a contravariant functor on χ.

We specified a concept of monetary value measures as a contravariant functor on χ. The resulting monetary value measures satisfy naturally so-called time consistency condition as well as dynamic programming principle.

Next, we showed a concrete shape of the largest Grothendieck topology for which monetary value measures satisfying given axioms become sheaves. By using sheafification functors, for any monetary value measure, we constructed its best approximation of the monetary value measure that satisfies given axioms in case the axioms are complete.

As a list of future's investigation, we will try to formulate a robust representation of concave monetary value measures within the category χ. We also seek the possibility to represent each individual axiom of monetary value measures as a specific Grothendieck topology which may give us an insight about different aspects of the axioms of monetary value measures. And then we will investigate the completeness condition against the important sets of axioms such as those of concave monetary value measures.

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GRADUATE SCHOOL OF INTERNATIONAL CORPORATE STRATEGY, HITOTSUBASHI UNIVERSITY, 2-1-2 HITOTSUBASHI, CHIYODA-KU, TOKYO, JAPAN

E-mail address: taka.adachi@gmail.com