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Hiroshi SASAKI

Graduate School of International Corporate Strategy,
Hitotsubashi University

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Abstract

In this paper, we provide a novel representation of delta-hedged option returns in a stochastic volatility environment. The representation of delta-hedged option returns in which we propose consists of two terms: volatility risk premium and parameter uncertainty. In an empirical analysis, we examine the delta-hedged option returns based on the historical simulation of a currency option market from October 2003 to June 2010. We find that the delta-hedged option returns for OTM put options are strongly affected by parameter uncertainty as well as the volatility risk premium, especially in the post-Lehman shock period.

Keywords: delta-hedged option returns, stochastic volatility, parameter uncertainty, volatility risk premium, currency option
JEL Classification: C, D, G

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†Graduate School of International Corporate Strategy, Hitotsubashi University, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8439, Japan, Tel.: 81 3 5219 2817, E-mail address : id10f002@g.hit-u.ac.jp

1 Introduction

In developing risk management strategies for financial option portfolios in incomplete markets, it is necessary to specify the sources of risks in the markets and to select a option pricing model which is consistent with those of specified risks. In particular, for the practitioners it is essential to consider the matters mentioned above for their risk management processes. But, even if the risk managers develop a sophisticated model with a consistent manner in terms of the specified risks, the option portfolios can not be hedged perfectly in incomplete markets due to the sources of unhedgeable risks and are exposed to the risk of significant losses in the processes of managing their option portfolios. Thus, the studies on empirical option prices and the features of delta-hedged option returns are important for the risk managers of option portfolios. Moreover, it is well known that delta-hedged option returns play a key role in identifying the principles of valuation and the sources of prices in actual option market.

For these reasons, a rich body of research on empirical option prices and delta-hedged option returns in financial option markets has developed in recent years with some stylized empirical analyses. Coval and Shumway[2001] examine expected option returns in the context of mainstream asset pricing theory and their results strongly suggest that something besides market risk is important in pricing the risk associated with option contracts. They imply that systematic stochastic volatility may be an important factor in pricing assets. Bakshi and Kapadia[2003] and Low and Zhang[2005] study delta-hedged option returns in a stock index option market and currency option markets respectively and they provide an evidence that expected delta-hedged option returns are not zero because of negative stochastic volatility risk premiums. Goyal and Saretto[2009] study a cross-section of stock option returns by sorting stocks on the difference between historical realized volatility and at-the-money implied volatility. They find that a zero-cost trading strategy that is long (short) in a position with a large positive (negative) difference between these two volatility measures produces an economically and statistically significant return due to some unknown risk factors or mispricing. Broadie, Chernov, and Johannes[2009] conclude that option portfolio returns can be well explained if we consider jump risk premiums or model parameter estimation risk. They assume that investors account for uncertainty in the spot volatility and parameters when pricing options.

Although these studies identify and investigate the sources of financial option prices in terms of some systematic risk factors or mispricing separately, they do not demonstrate any relative contribution to option prices between systematic risk factors and mispricing based on a unified approach.

Jones[2006] presents the most recent research that provides a unified approach to demonstrate the relative contribution of the sources of stock index option prices based on a non-linear factor analysis. He examines the historical performances of equity index option portfolios in the period from January 1986 to September 2000 and shows that priced risk factors such as stochastic volatility and jump contribute to their extraordinary average returns but are insufficient to explain their magnitudes, particularly for short-term out-of-the-money puts. This may be the only study that provides a unified approach to demonstrate the relative contribution of the sources of financial option prices based on a stylized model, but the author does not reveal any sources besides the priced risk factors such as stochastic volatility and jump that contribute to the option portfolios' extraordinary average returns. The author also does not show the time dependency of the relative contribution between the systematic risk factors and other potential sources such as mispricing to financial option prices especially during the period of the recent financial crisis in 2008 because his empirical analysis is based on the period from January 1986 to September 2000.

In this papaer, we present the relative contribution analysis of the effect of systematic risk factors and the effect of "parameter uncertainty" of option valuation models on financial option prices based on a historical simulation in the pre- and post Lehman crisis period. Theoretical models often assume that the economic agent who makes an optimal financial decision knows the true parameters of the model. But the true parameters are rarely if ever known to the decision maker. In reality, model parameters have to be estimated based on historical information and, hence, the model's usefulness depends partly on how good the estimates are. This gives rise to estimation risk in virtually all option valuation models.

We provide a novel representation of delta-hedged option returns in a stochastic volatility environment. The representation of delta-hedged option returns provided in this paper consists of two terms; volatility risk premium and parameter uncertainty. In an empirical analysis, we examine the delta-hedged option returns of the USD-JPY currency options based on a historical simulation from October 2003 to June 2010. We find that the delta-hedged option returns for OTM put options are strongly affected by parameter uncertainty as well as the volatility risk premium, especially in the post-Lehman shock period.

To the best of our knowledge, this is the first empirical research on the relative contribution analysis of the effects of systematic risk factors and parameter uncertainty on delta-hedged option returns in a stochastic volatility environment. Of course, there are some prior studies on the effects of parameter (or model) uncertainty on pricing financial options (Green and Figlewski[1999], Bunnin, Guo, and Ren[2002], Cont[2006], etc.) , but there is no empirical evidence that shows the relative contribution of parameter uncertainty to delta-hedged option returns or the time dependency of that contribution. In our empirical study, approximately 13 % of the value of the OTM currency option premium is generated by the existence of parameter uncertainty in the post-Lehman crisis period, and this effect induced by parameter uncertainty on option prices is more significant than the effect of the volatility risk premium. One of the most important implications of our study is that the sign and the level of the expected delta-hedged option returns do not necessarily explain the existence of volatility risk premiums. An important point to emphasize is that there may be additional important factors such as parameter uncertainty that make an impact on delta-hedged option returns, rendering standard hedging-based tests on volatility risk premiums explored and examined by, for example, Bakshi and Kapadia[2003] and Low and Zhang[2005], unreliable.

The paper is organized as follows: Section 2 describes the model structure and provides the explicit representation of delta-hedged option returns. An estimation methodology for the time-varying volatility risk premium in the USD-JPY currency option market is also explored in this section. Section 3 describes the basic methodology used in our empirical analysis, and Section 4 illustrates the nature of the delta-hedged option returns and presents empirical findings on the relative contributions of the effects of the volatility risk premium and parameter uncertainty on delta-hedged option returns. Section 5 summarizes the main results of the paper.

2 The Model and the Methodology

2.1 An explicit representation for delta-hedged option returns

We start with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\geq 0})$, $t \in [0, T]$ and consider a two dimensional exchange rate process that allows return volatility to be stochastic under the physical probability measure $\mathbb{P}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + \sigma_t \sqrt{1-\rho_t^2} dW_t^1 + \sigma_t \rho_t dW_t^2, \\ d\sigma_t &= \theta_t dt + \eta_t dW_t^2, \end{aligned} \tag{1}$$

where μ_t, θ_t, η_t and ρ_t are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic processes which allow the above equations to have a strong solution and these processes are independent of S_t. (W_t^1, W_t^2, W_t^3) denote a standard 3-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we give the information set $\mathcal{F}_t$ as a sigma-algebra of $\sigma\{W_s^1, W_s^2, W_s^3 | s \leq t\} \vee \mathcal{N}$ where $\mathcal{N}$ is the null set. In this paper, we assume that the price and volatility process represented by (1) are observable in the financial market and, to avoid complexity, hereinafter we also assume that the drift parameter μ_t in (1) is given by $r_d - r_f$, where $r_d \in \mathbb{R}$ and $r_f \in \mathbb{R}$ denote the domestic and the foreign risk-free interest rate, respectively.

We limit the model of an exchange rate process to a stochastic volatility model and do not consider other factors such as the jump. Although this model is rather restrictive, but in Andersen, Bollerslev, Diebold and Labys[2000], based on ten years of high-frequency returns for the Deutschemark - U.S. Dollar and Japanese Yen - U.S. Dollar exchange, they provide indirect support for the assertion of a jumpless diffusion with a fact that the presence of jumps is likely to result in a violation of the empirical normality of the standardized returns.

It is well known that the absence of arbitrage opportunities is essentially equivalent to the existence of a probability measure $\mathbb{Q}$, equivalent to the physical probability measure $\mathbb{P}$, under which the discounted prices process is an $\mathcal{F}_t$-adapted martingale; such a probability will be called equivalent martingale measure. Any equivalent martingale measure $\mathbb{Q}$ is characterized by a continuous version of its density process with respect to $\mathbb{P}$ which can be written from the integral form of martingale representation

$$M_t \equiv \frac{d\mathbb{Q}}{d\mathbb{P}} \,|_{\mathcal{F}_t} \equiv \exp\Big(- \int_0^t \nu_u dW_u^1 - \int_0^t \lambda_u dW_u^2 - \frac{1}{2}\int_0^t \nu_u^2 du - \frac{1}{2}\int_0^t \lambda_u^2 du \Big),$$

where (ν_t, λ_t) is adapted to $\mathcal{F}_t$ and satisfies the integrability conditions $\int_0^T \nu_u^2 du < \infty$ and $\int_0^T \lambda_u^2 du < \infty$ a.s.. Two processes of ν_t and λ_t are interpreted as the price of risk premia relative respectively to the two sources of uncertainty W_t^1 and W_t^2. In particular, if $\Lambda_t \equiv M_t B_t$ denotes the discount factor process where $B_t \equiv \exp\big(- r_d t\big)$ and $r_d \in \mathbb{R}$ is the domestic risk-free interest rate, then the price of volatility risk λ_t is represented as $\lambda_t \equiv -\mathrm{Cov}_t(\frac{d\Lambda_t}{\Lambda_t}, d\sigma_t)$ (See, e.g., Cochrane[2005]) and a positive correlation between the discount factor process Λ_t and the volatility process σ_t implies a negative λ_t. To understand clearly, for example, if we could assume the stochastic volatility model proposed by Heston[1993] for (1) and the representative agent with a power utility function in the financial market, then we can derive the following equation [1]

$$\mathrm{Cov}_t\Big(\frac{d\Lambda_t}{\Lambda_t}, d\sigma_t\Big) = -\gamma \rho v \sigma_t,$$

where $\gamma \in \mathbb{R}$ is the risk aversion parameter for the representative agent and $\rho \in [-1, 1]$ and $v > 0$ correspond to ρ_t and η_t respectively in (1). This relation suggests that the market price of volatility risk λ_t is proportional to ρ, and if the correlation between volatility changes and changes in the exchange rate is negative, then the market price of volatility risk is also negative. Hereinafter we also describe $\mathbb{Q}$ by $\mathbb{Q}[\lambda_t]$ to emphasize that $\mathbb{Q}$ is depend on the process of λ_t.

$C_t^{TP} \equiv F(t, S_t, \sigma_t)$ denote the time-t theoretical price of an European-type call option [2] which is consistent with the equation (1) and the market preference. This C_t^{TP} is written on S_t, struck at K, expiring at time T, and represented by a $C^{1,2,2}$-function $F(t, S, \sigma)$. Under the equivalent martingale measure $\mathbb{Q}$ (which is also consistent with (1) and the market preference), we can give an explicit representation for C_t^{TP} in the following manner:

$$C_t^{TP} \equiv F(t, S_t, \sigma_t) = e^{-r_d(T-t)} \mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0) \mid \mathcal{F}_t]. \tag{2}$$

[1] Details will be described in section 4.1.

[2] In this section, we focus on an European call option, but the discussion and results provided in this section can apply more generally to other options, that is, put options or straddle options.

Let us explore the theoretical delta-hedged option returns for $C_t^{TP} = F(t, S_t, \sigma_t)$ based on the above assumption.

Under the assumption of $0 \le \tau \le T - t$, we can derive a following equation via Ito's lemma:

$$
\begin{aligned}
C_{t+\tau}^{TP} = C_t^{TP} &+ \int_t^{t+\tau} \frac{\partial F}{\partial S}(u, S_u, \sigma_u) dS_u \\
&+ \int_t^{t+\tau} \frac{\partial F}{\partial \sigma}(u, S_u, \sigma_u) d\sigma_u + \int_t^{t+\tau} \mathcal{D}F(u, S_u, \sigma_u) du,
\end{aligned} \tag{3}
$$

where

$$
\begin{aligned}
\mathcal{D}F(t, S_t, \sigma_t) = \frac{\partial F}{\partial t}(t, S_t, \sigma_t) &+ \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 F}{\partial S^2}(t, S_t, \sigma_t) \\
&+ \frac{1}{2}\eta_t^2 \frac{\partial^2 F}{\partial \sigma^2}(t, S_t, \sigma_t) + \rho_t \eta_t \sigma_t S_t \frac{\partial^2 F}{\partial S \partial \sigma}(t, S_t, \sigma_t).
\end{aligned}
$$

If we define the delta-hedged gain and loss (hereinafter, DHGL) $\Pi_{t,t+\tau}$ for C_t^{TP} in the period of $[t, t+\tau]$ by

$$
\begin{aligned}
\Pi_{t,t+\tau} \equiv C_{t+\tau}^{TP} - C_t^{TP} &- \int_t^{t+\tau} \frac{\partial F}{\partial S}(u, S_u, \sigma_u) dS_u \\
&- \int_t^{t+\tau} \Big(r_d F(u, S_u, \sigma_u) - (r_d - r_f) S_u \frac{\partial F}{\partial S}(u, S_u, \sigma_u) \Big) du,
\end{aligned}
$$

then, from (3) it follows

$$
\begin{aligned}
\Pi_{t,t+\tau} &= \int_t^{t+\tau} \frac{\partial F}{\partial \sigma}(u, S_u, \sigma_u) d\sigma_u + \int_t^{t+\tau} \mathcal{D}F(u, S_u, \sigma_u) du \\
&\quad - \int_t^{t+\tau} \Big(r_d F(u, S_u, \sigma_u) - (r_d - r_f) S_u \frac{\partial F}{\partial S}(u, S_u, \sigma_u) \Big) du \\
&= \int_t^{t+\tau} \mathcal{L}F(u, S_u, \sigma_u) du + \int_t^{t+\tau} \eta_u \frac{\partial F}{\partial \sigma}(u, S_u, \sigma_u) dW_u^2,
\end{aligned} \tag{4}
$$

where

$$
\begin{aligned}
\mathcal{L}F(u, S_u, \sigma_u) = \theta_u &\frac{\partial F}{\partial \sigma}(u, S_u, \sigma_u) \\
&+ \mathcal{D}F(u, S_u, \sigma_u) - \Big[r_d F(u, S_u, \sigma_u) - (r_d - r_f) S_u \frac{\partial F}{\partial S}(u, S_u, \sigma_u) \Big].
\end{aligned} \tag{5}
$$

All the discussion on the DHGL explored above is valid when we know all the parameters of the model (1) perfectly and price options by a consistent manner with the model (1) under the known parameters. However, the option market participants could not necessarily price options based on the true equation (1) because they essentially could not know the true parameters of (1) due to limited information or might misestimate the set of parameters due to parameter uncertainty. That is to say that the option market participants might believe the another set of parameters that are not necessarily true while they try to estimate the true dynamics (1) based on historical market information. In this paper we assume that the available information set for the representative option market maker at time t is given as a sigma-algebra $\mathcal{G}_t \equiv \sigma\{W_s^1, W_s^2 | s \leq t\} \vee \mathcal{N}$ and suppose that the representative option market maker estimates the another set of parameters $\tilde{\theta}_t \equiv \mathbb{E}[\theta_t \mid \mathcal{G}_t]$, $\tilde{\eta}_t \equiv \mathbb{E}[\eta_t \mid \mathcal{G}_t]$, and $\tilde{\rho}_t \equiv \mathbb{E}[\rho_t \mid \mathcal{G}_t]$ for the drift of the volatility process, the diffusion coefficient of the volatility process, and the correlation between volatility changes and changes in the exchange rate, respectively. The parameters of $\tilde{\theta}_t$, $\tilde{\eta}_t$, and $\tilde{\rho}_t$ may not be equal to the parameters of θ_t, η_t, and ρ_t, respectively, because θ_t, η_t, and ρ_t are not necessarily adapted to the filtration $\{\mathcal{G}_t\}_{t\geq 0}$ (They are adapted only to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$). Under such condition, it is natural to assume that the representative option market maker prices options based on the following another model for an exchange rate process $\tilde{S}_t$ [3] :

$$\begin{aligned} \frac{d\tilde{S}_t}{\tilde{S}_t} &= \mu_t dt + \tilde{\sigma}_t \sqrt{1-\tilde{\rho}_t^2} dW_t^1 + \tilde{\sigma}_t \tilde{\rho}_t dW_t^2, \\ d\tilde{\sigma}_t &= \tilde{\theta}_t dt + \tilde{\eta}_t dW_t^2. \end{aligned} \tag{6}$$

The quoted option price at time t determined by the representative option market maker which is consistent with the model (6) is denoted by

$$C_t^M \equiv G(t, \tilde{S}_t, \tilde{\sigma}_t) \equiv e^{-r_d(T-t)} \mathbb{E}^{\mathbb{Q}}[\max(\tilde{S}_T - K, 0) \mid \mathcal{G}_t],$$

[4] where G is a $C^{1,2,2}$-function that satisfies the following pricing equation:

$$\begin{aligned} \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 G}{\partial \tilde{S}^2}(t, \tilde{S}_t, \tilde{\sigma}_t) &+ \frac{1}{2}\tilde{\eta}_u^2 \frac{\partial^2 G}{\partial \tilde{\sigma}^2}(t, \tilde{S}_t, \tilde{\sigma}_t) \\ &+ \tilde{\rho}_t \tilde{\eta}_t \sigma_t S_t \frac{\partial^2 G}{\partial \tilde{S} \partial \tilde{\sigma}}(t, \tilde{S}_t, \tilde{\sigma}_t) + (r_d - r_f) S_t \frac{\partial G}{\partial \tilde{S}}(t, \tilde{S}_t, \tilde{\sigma}_t) \\ &+ (\tilde{\theta}_t - \lambda_t[\sigma]) \frac{\partial G}{\partial \tilde{\sigma}}(t, \tilde{S}_t, \tilde{\sigma}_t) + \frac{\partial G}{\partial t}(t, \tilde{S}_t, \tilde{\sigma}_t) - r_d G(t, \tilde{S}_t, \tilde{\sigma}_t) = 0. \end{aligned} \tag{7}$$

Due to such parameter uncertainty on the volatility dynamics introduced above, the option market participants try to hedge according to not the true function F but the misspecified function G, and this fact leads to the following proposition:

Proposition 1 *If the representative option market maker tries to hedge based on the misspecified function G, then the expectation of the delta-hedged gain and loss from time t to $t+\tau$, $\Pi_{t,t+\tau}^G$, for the market maker, that is, $\mathbb{E}^{\mathbb{P}}[\Pi_{t,t+\tau}^G]$, is represented by the following*

[3]We assume that the SDE of (6) has a strong solution for $\tilde{S}_t$ under the parameters of $\tilde{\theta}_t$, $\tilde{\eta}_t$, and $\tilde{\rho}_t$.
[4]The notation of C_t^M (or C_t^{TP}) is in line with Galai[1983].

formula:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\Pi^G_{t,t+\tau}\big] = &\int_t^{t+\tau} \mathbb{E}^{\mathbb{P}}\Big[\frac{1}{2}(\eta_u^2 - \tilde{\eta}_u^2)\frac{\partial^2 G}{\partial \tilde{\sigma}^2}(u, \tilde{S}_u, \tilde{\sigma}_u) \\ &+ (\rho_u \eta_u - \tilde{\rho}_u \tilde{\eta}_u)\sigma_u S_u \frac{\partial^2 G}{\partial \tilde{S} \partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u) + (\theta_u - \tilde{\theta}_u)\frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u)\Big] du \\ &+ \int_t^{t+\tau} \mathbb{E}^{\mathbb{P}}\Big[\lambda_u[\sigma]\frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u)\Big] du.\end{aligned} \tag{8}$$

Proof *See the Appendix.* $\square$

An interpretation for the expected DHGL (8) is given as follows. If the price process of an underlying exchange rate assumed by the representative option market maker coincide with the true price process, that is to say, $\theta_t = \tilde{\theta}_t$, $\eta_t = \tilde{\eta}_t$ and $\rho_t = \tilde{\rho}_t$, then the first term in the right side of (8) equals zero and the expected DHGL under the physical measure $\mathbb{P}$ will be represented as

$$\mathbb{E}^{\mathbb{P}}\big[\Pi^G_{t,t+\tau}\big] = \int_t^{t+\tau} \mathbb{E}^{\mathbb{P}}\big[\lambda_u \frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u)\big] du.$$

Thus, the expected DHGL will be perfectly determined only by the level of the volatility risk premium λ_t. In addition to the above assumption, if we assume a linear form of $\lambda_u \equiv \lambda \sigma_u$ $(\lambda \in \mathbb{R}^1)$ for the volatility risk premium as well as the stochastic volatility model proposed by Heston[1993], the sign of the expected DHGL will coincide with the sign of the volatility risk premium parameter λ because $\sigma_u > 0$ and $\frac{\partial C_u}{\partial \tilde{\sigma}_u} > 0$. But, if the parameters for the representative option market maker, $\tilde{\theta}_t$, $\tilde{\eta}_t$ and $\tilde{\rho}_t$, are not equal with those of true parameters in (1) (i.e. θ_t, η_t and ρ_t), the sign and the magnitude of the expected DHGL will be affected by the parameter uncertainty term represented as the first term in the right side of (8) as well as the volatility risk premium term.

Let us provide a proposition related to the second term in the right side of (8) for the purpose of obtaining a more testable representation of the expected DHGL.

Proposition 2 *If $C_t^M[\lambda_u]$ denotes the time-t call option price which is consistent with the underlying exchange rate process (6) and the equivalent martingale measure $\mathbb{Q}[\lambda_u]$, the following inequation can be derived:*

$$C_t^M[0] - C_t^M[\lambda_u] \leq (\geq) \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{\sigma}_u}\Big] du \leq (\geq) \Big(1 + r_d(T-t)\Big)\Big(C_t^M[0] - C_t^M[\lambda_u]\Big),$$

where $\lambda_u \geq (\leq) 0$. Moreover, if we are able to assume $r_d \approx 0$, then this inequation can be further simplified to the approximation presented below:

$$C_t^M[0] - C_t^M[\lambda_u] \approx \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{\sigma}_u}\Big] du. \tag{9}$$

Proof *See the Appendix.* $\square$

Under the zero interest rate financial policy in Japan after the year of 2000, we are allowed to consider r_d to be approximately zero. So we can assume that (9) will be well-suited for the exchange rates whose base currency is the Japanese Yen.

Substituting the approximation of (9) for (8), an explicit representation of expected DHGL for the representative option market maker can be derived as follows:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\Pi^G_{t,t+\tau}\right] \approx \int_t^{t+\tau} & \mathbb{E}^{\mathbb{P}}\Big[\frac{1}{2}(\eta_u^2 - \tilde{\eta}_u^2)\frac{\partial^2 G}{\partial \tilde{\sigma}^2}(u, \tilde{S}_u, \tilde{\sigma}_u) \\ & + (\rho_u \eta_u - \tilde{\rho}_u \tilde{\eta}_u)\sigma_u S_u \frac{\partial^2 G}{\partial \tilde{S} \partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u) + (\theta_u - \tilde{\theta}_u)\frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u)\Big] du \\ & + C_t^M[0] - C_t^M[\lambda_u].\end{aligned} \tag{10}$$

We will simulate the DHGLs with historical market data and estimate the left side of (10) empirically in the following section. So if we can valuate the second term in the right side of (10) in each period, we will be able to provide a contribution analysis of the effect of parameter uncertainty and the effect of the volatility risk premium on the expected DHGL based on (10) empirically. In this paper, we assume that the representative option market maker determines option market prices being consistent with (6) by estimating the set of parameters in (6), that is to say, $\tilde{\theta}_t$, $\tilde{\eta}_t$, and $\tilde{\rho}_t$, based on historical market price data. Thus under the assumption stated above, if we can estimate the set of parameters in (6) with historical market price data and calibrate the volatility risk premium implied in option market prices in each period, we will be able to provide a contribution analysis on the expected DHGL explicitly. An estimation strategy for $\tilde{\theta}_t$, $\tilde{\eta}_t$, and $\tilde{\rho}_t$ will be described in the next section. In the next subsection, we provide a calibration methodology for the volatility risk premium parameter implied in option market prices in each period with estimated parameters of $\tilde{\theta}_t$, $\tilde{\eta}_t$, and $\tilde{\rho}_t$.

Hereinafter we assume $\theta_t \equiv -k\sigma_t$, $\eta_t \equiv v$, $\rho_t \equiv \rho$, $\tilde{\theta}_t \equiv -\tilde{k}\sigma_t$, $\tilde{\eta}_t \equiv \tilde{v}$, and $\tilde{\rho}_t \equiv \tilde{\rho}$ (k, v, ρ, $\tilde{k}$, $\tilde{v}$, and $\tilde{\rho}$ are constants), that is to say, the stochastic volatility model proposed by Heston[1993] and investigate the expected DHGL represented by (10) in detail under such assumptions. Heston[1993] also assumes linear form of $\lambda_t[\sigma] \equiv \lambda\sigma_t$ for the volatility risk premium and we are also assume that form in line with Heston[1993].

2.2 Estimation for the Volatility Risk Premium

As mentioned above, we assume that the representative option market maker prices options based on the model under the assumptions of $\tilde{\theta}_t \equiv -\tilde{k}\sigma_t$, $\tilde{\eta}_t \equiv \tilde{v}$, and $\tilde{\rho}_t \equiv \tilde{\rho}$ in (6), or

$$\begin{aligned}\frac{d\tilde{S}_t}{\tilde{S}_t} &= (r_d - r_f)dt + \tilde{\sigma}_t\sqrt{1-\tilde{\rho}^2}dW_t^1 + \tilde{\sigma}_t\tilde{\rho}dW_t^2, \\ d\tilde{\sigma}_t &= -\tilde{k}\tilde{\sigma}_t dt + \tilde{v}dW_t^2.\end{aligned} \tag{11}$$

If we assume the formula of the volatility risk premium as $\lambda_t \equiv \lambda\tilde{\sigma}_t$ ($\lambda \in \mathbb{R}$), that is to say, a linear form on the volatility, (11) will be rewrited as below under the risk neutral measure $\mathbb{Q}[\lambda\tilde{\sigma}_t]$,

$$\begin{aligned}\frac{d\tilde{S}_t}{\tilde{S}_t} &= (r_d - r_f)dt + \tilde{\sigma}_t\sqrt{1-\tilde{\rho}^2}d\tilde{W}_t^1 + \tilde{\sigma}_t\tilde{\rho}d\tilde{W}_t^2, \\ d\tilde{\sigma}_t &= -(\tilde{k}+\lambda)\tilde{\sigma}_t dt + \tilde{v}d\tilde{W}_t^2,\end{aligned} \tag{12}$$

where $\tilde{W}_t \equiv \left(\tilde{W}_t^1, \tilde{W}_t^2\right)^t$ is two-dimensional Brownian motion under $\mathbb{Q}[\lambda\tilde{\sigma}_t]$ whose each component is represented as below:

$$\tilde{W}_t^1 \equiv W_t^1 + \int_0^t \nu_u du \quad \text{and} \quad \tilde{W}_t^2 \equiv W_t^2 + \lambda\int_0^t \tilde{\sigma}_u du.$$

Under (12), we can derive the expectation of instantaneous variance at time t under $\mathbb{Q} \equiv \mathbb{Q}[\lambda\tilde{\sigma}_t]$,

$$\mathbb{E}_t^{\mathbb{Q}}[\tilde{\sigma}_u^2] = \tilde{\sigma}_t^2\exp(-2(\tilde{k}+\lambda)(u-t)) + \frac{\tilde{v}^2}{2(\tilde{k}+\lambda)}\Big(1-\exp(-2(\tilde{k}+\lambda)(u-t))\Big), \quad t \le u.$$

Thus the expectation of realized variance $RV_{t,T}$ in the period of $[t,T]$ under $\mathbb{Q}$ will be represented as below,

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}[RV_{t,T}] &= \mathbb{E}_t^{\mathbb{Q}}[\frac{1}{T-t}\int_t^T \tilde{\sigma}_u^2 du] = \frac{1}{T-t}\int_t^T \mathbb{E}_t^{\mathbb{Q}}[\tilde{\sigma}_u^2]du \\ &= \frac{\tilde{v}^2}{2(\tilde{k}+\lambda)} + \frac{\exp(-2(\tilde{k}+\lambda)T) - \exp(-2(\tilde{k}+\lambda)t)}{2(\tilde{k}+\lambda)(T-t)}\Big(\frac{\tilde{v}^2}{2(\tilde{k}+\lambda)} - \tilde{\sigma}_t^2\Big).\end{aligned} \tag{13}$$

On the other hand, Carr and Wu[2009] provide the formula for the risk neutral expected value of return variance which can be well approximated with the value of a particular portfolio of options [5].

Proposition 3 (Carr and Wu(2009)) *Under no arbitrage, the time-t risk-neutral expected value of the return quadratic variation of an asset over horizon $[t,T]$ can be approximated by the continuum of European out-of-the-money option prices across all strikes*

[5] Carr and Wu[2009] assume that the futures price F_t solves the following stochastic differential equation,

$$dF_t = F_{t-}\sigma_{t-}dW_t + \int_{(-\infty,\infty)\backslash 0} F_{t-}\left(e^x - 1\right)\left[\mu(dx,dt) - \nu_t(x)dxdt\right]$$

(see Carr and Wu[2009] for details on a notation). The equation represented above models the futures price change as the summation of the increments of two orthogonal martingales: a purely continuous martingale and a purely discontinuous (jump) martingale. This decomposition is generic for any continuous time martingales. So, in general, Proposition 2 should be stated including the effect of jump component. But, in this paper, we only assume a *continuous* martingale in order to represent an underlying exchange rate process, so we leave the term induced by the jump component out of (14) in Proposition 2.

$K > 0$ and at the same maturity date T

$$\mathbb{E}_t^{\mathbb{Q}}[RV_{t,T}] = \frac{2}{T-t}\int_0^\infty \frac{\Theta_t(K,T)}{B_t(T)K^2}dK, \tag{14}$$

where $B_t(T)$ denotes the time-t price of a bond paying one dollar at T, $\Theta_t(K,T)$ denotes the time-t value of an out-of-the-money option with strike price $K > 0$ and maturity $T \geq t$ (a call option when $K > F_t$ and a put option when $K \leq F_t$).

Proof *See proof of Proposition1 in Carr and Wu[2009].* □

Using the set of parameters in (6) and option prices $\Theta_t(K,T)$ quoted in an option market, we can explicitly estimate λ based on the following equation

$$\begin{aligned}\frac{2}{T-t}\int_0^\infty \frac{\Theta_t(K,T)}{B_t(T)K^2}dK = &\frac{\tilde{v}^2}{2(\tilde{k}+\lambda)} \\ &+ \frac{\exp(-2(\tilde{k}+\lambda)T)-\exp(-2(\tilde{k}+\lambda)t)}{2(\tilde{k}+\lambda)(T-t)}\Big(\frac{\tilde{v}^2}{2(\tilde{k}+\lambda)}-\tilde{\sigma}_t^2\Big),\end{aligned} \tag{15}$$

which can be derived with (13) and (14). Thus, using the λ estimated with the equation (15), we can calculate $C_t^M[0]-C_t^M[\lambda_u]$, the second term in the right hand side of (10), at each time t using the explicit closed formula for the European-type call option proposed by Heston[1993]:

$$\begin{aligned}
&C_t^M[\lambda\tilde{\sigma}_t] = S_tP_1 + e^{-r\tau}KP_2,\\
&\quad \textit{where}\\
&\qquad P_j = \frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-\sqrt{-1}\phi ln(K)}F_j}{\sqrt{-1}\phi}\right]d\phi,\\
&\qquad F_j = e^{C+D\sigma_t^2+\sqrt{-1}\phi ln(S_t)},\\
&\qquad C = (r_d-r_f)\tau\phi\sqrt{-1}+\frac{1}{4}\left[(\beta_j-2\tilde{\rho}\tilde{v}\phi\sqrt{-1}+h)\tau-2ln\Big(\frac{1-ge^{h\tau}}{1-g}\Big)\right],\\
&\qquad D = \frac{\beta_j-2\tilde{\rho}\tilde{v}\phi\sqrt{-1}+h}{4\tilde{v}^2}\Big(\frac{1-e^{h\tau}}{1-ge^{h\tau}}\Big),\\
&\qquad g = \frac{\beta_j-2\tilde{\rho}\tilde{v}\phi\sqrt{-1}+h}{\beta_j-2\tilde{\rho}\tilde{v}\phi\sqrt{-1}-h},\\
&\qquad h = \sqrt{(2\tilde{\rho}\tilde{v}\phi\sqrt{-1}-\beta_j)^2-4\tilde{v}^2(2u_j\phi\sqrt{-1}-\phi^2)}, \qquad (j=1,2)\\
&\quad \textit{and}\\
&\qquad \tau = T-t, u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, \beta_1 = 2\tilde{k}+\lambda-2\tilde{\rho}\tilde{v}, \beta_2 = 2\tilde{k}+\lambda.
\end{aligned} \tag{16}$$

Needless to say, we can also derive the closed formula for the European-type put option by using (16) and the put-call parity relation and will use these closed formulas for pricing currency options in the following empirical simulations.

3 Data and Methodology for an Empirical Implementation

3.1 Description of the OTC Currency Option Market and Data

In our empirical study, we examine the expected DHGL and its contribution analysis with the USD-JPY spot exchange rate and the USD-JPY currency options with maturities of one month traded on the OTC market. The OTC currency option market has some special features and conventions. First, option prices in the OTC market are quoted in terms of deltas and implied volatilities instead of strikes and money prices, as in the organized option exchanges. At the time of settling a given deal, the implied volatility quotes are translated to money prices using the Garman-Kohlhagen formula, which is the equivalent of the Black-Scholes formula for currency options. This arrangement is convenient for option dealers in that they do not have to change their quotes every time the spot exchange rate moves. However, it is important to note that this does not mean that option dealers necessarily believe that the Black-Scholes assumptions are valid. They use the formula only as a one-to-one nonlinear mapping between the volatility delta space (where the quotes are made) and the strike premium space (in which the final specification of the deal is expressed for the settlement). Second, most transactions in the market involve option combinations. The popular combinations are straddles, risk reversals, and strangles. Among these, the most liquid combination is the standard delta-neutral straddle contract, which is a combination of a call and a put with the same strike. The strike price is set, together with the quoted implied volatility space, such that the delta of the straddle computed on the basis of the Garman-Kohlhagen formula is zero.

Because the standard straddle is by design delta neutral on the deal date, its price is not sensitive to the market price of the underlying foreign currency. However, it is sensitive to changes in volatility. Because of its sensitivity to volatility risk, delta-neutral straddles are widely used by participants in the OTC market to hedge and trade volatility risk. If the volatility risk is priced in the OTC market, then delta-neutral straddles are the best instruments through which to observe the risk premium. For this reason, Cocal and Shumway[2001] use delta-neutral straddles in their empirical study of expected returns on equity index options and find that a volatility risk premium is priced in the equity index option market.

We use the WM/Reuter closing spot rate for the exchange rate data, the LIBOR 1M interest rates for the domestic (Japan) and the foreign (United States of America) interest rates, and quoted implied volatility data from Bloomberg. The implied volatility data is from the European type put and call OTC currency options with maturities of one month and strike prices of 5 delta, 10 delta, 15 delta, 25 delta, 35 delta and ATM, respectively. In the following empirical simulation, we price the options using bid prices quoted in the actual market at each time point in order to take account of transactions costs when simulating the profit and loss generated by a delta-neutral hedging strategy with a short position of the European option. Our data sample starts in October 2003 (because of data availability of the implied volatility in the USD-JPY currency option market) and ends in June 2010.

Fig.3, Fig.4 and Fig.5 show the time series data for the USD-JPY WM/Reuter closing spot rate, the ATM implied volatility for the USD-JPY European put option, and the mid-bid price spread of the ATM implied volatility which indicates the level of transactions costs for selling strategies of the European ATM options at each time point, respectively. Table 7 provides descriptive statistics for the implied volatilities in the period from October 2003 to June 2010. In this table we can see the feature of "volatility skew", which indicates that the implied volatilities for OTM puts are higher than those for OTM calls during the period under consideration.

3.2 Parameter Estimation for the Heston[1993] Stochastic Volatility Model

In this paper, we estimate a set of parameters for the Heston[1993] stochastic volatility model specified in equation (6) with the maximum likelihood method proposed by Aït-Sahalia[2001] and Aït-Sahalia and Kimmel[2007]. Aït-Sahalia and Kimmel[2007] provide an approximation formula for the likelihood function using the Hermite series expansion of the transition probability density of the Heston[1993] stochastic volatility model and propose a methodology for estimating the parameters of multivariate diffusion processes via the maximum likelihood method with discrete-sampled price data. They derive a closed form likelihood function used explicitly to estimate parameters of a two-dimensional diffusion process consisting of an underlying asset price and its instantaneous volatility or the option price associated with it. In this study, we use a historical 20-day realized volatility as a proxy for the instantaneous volatility and estimate the model parameters based on the maximum likelihood method proposed by Aït-Sahalia and Kimmel[2007]. We update the model parameters daily using the historical data of 1,750 days with a rolling estimation procedure.

3.3 Estimation of the Volatility Risk Premium

To estimate the volatility risk premium parameter λ with equation (15), we need to calculate the integral term in that equation by a discretization of that integral. As mentioned in the previous subsection, we have only a grid of 11 implied volatility points in terms of the strike price, so that we first interpolate the implied volatilities at different moneyness levels with the polynomial approximation methodology proposed by Brunner and Hafner[2003] to obtain a fine curve of implied volatilities [6]. Then we obtain the value of the integral in (15) by applying the numerical integral technique. We calculate $B_t(T)$ in (15) as $\exp(-r_d(T-t))$, which is a zero coupon bond price whose maturity date is T.

3.4 Estimation of the Expected DHGL

We employ an empirical analysis based on a historical simulation to estimate the magnitude of the expected DHGL for the delta-hedged option strategy with a short position of the one-month ATM-forward straddle [7] or the one-month OTM delta-25 put. In particular, beginning on our simulation on the date of October 31, 2003, we compute the DHGLs for those of the strategies at the end date after one month and repeat the same computations on the following day after October 31, 2003. The final simulation starts on May 31, 2010 and ends at June 30, 2010. We finally collect the DHGL results for 1,717 samples for each delta-hedged option strategy through those iterated simulations. We employ the Garman-Kohlhagen model, as an extension of the Black-Scholes model to currency options, to compute the delta of the short option positions for tractability, even though the delta computed from the Garman-Kohlhagen model may differ from the delta computed from a stochastic volatility model. If we use C to denote the European call option price on an exchange rate, P as the European put option price, S_0 as the spot rate level on that exchange rate, r_f as the foreign risk free rate, r_d as the domestic risk free rate, σ as the volatility, T as the maturity, and $N(\cdot)$ as the cumulative standard normal distribution function, Garman-Kohlhagen[1983] provides the closed formula for the prices of European currency options as follows:

$$C = S_0 e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2) \quad , \quad P = K e^{-r_d T} N(-d_2) - S_0 e^{-r_f T} N(-d_1),$$

where

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r_d - r_f + \frac{\sigma^2}{2})}{\sigma\sqrt{T}} \quad , \quad d_2 = \frac{ln(\frac{S_0}{K}) + (r_d - r_f - \frac{\sigma^2}{2})}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}. \tag{17}$$

If the prices of European currency options are represented as described above, the delta values of call and put options are computed with the following respective formulas

$$\triangle_{call} = e^{-r_f T} N(d_1) \quad , \quad \triangle_{put} = e^{-r_f T} \big(N(d_1) - 1\big).$$

[6] Brunner and hafner[2003] approximate the implied volatility $\sigma_t^T(K)$, whose the strike price is K and the maturity date is T, as a polynomial as follows; $\sigma_t^T(K) = \beta_0 + \beta_1 M + \beta_2 M^2 + D\beta_3 M^3$, where $D \equiv 0(if M \leq 0), \equiv 1(if M > 0)$ and $M \equiv \log\big(\frac{K}{F_t^T}\big)/\sqrt{T-t}$, under the definition that F_t^T is a forward rate whose the maturity date is T at each time t.

[7] The ATM-forward straddle contract is a combination of a call and a put option with the same strike price of ATM forward rate and the same maturity to the underlying forward contract.

Bakshi and Kapadia[2003] and Low and Zhang[2005] provide a simulation exercise that shows using the Black-Scholes delta-hedge ratio instead of the stochastic volatility counterpart has a negligible effect on the DHGL results, and they insist that the empirical analysis results regarding the existence and the sign of the volatility risk premium would not be affected by the decision regardless of which model is used to compute the delta. We also use the Garman-Kohlhagen model to compute the delta in line with the studies of Bakshi and Kapadia[2003] and Low and Zhang[2005]. In our delta-neutral hedge simulations, the volatility σ in the equation (17) is estimated using historical daily return data of 20 days.

We rebalance the delta-hedged option portfolio daily and measure the DHGL $\Pi^G_{t,T}$ in the period from the contract date t to the maturity date T using the following formula:

$$\Pi^G_{t,T} = C^M_t - C^M_T - \sum_{n=0}^{N-1} \Delta_{t_n}(S_{t_n} - S_{t_{n+1}}) + \sum_{n=0}^{N-1}(r_d C^M_t - (r_d - r_f)\Delta_{t_n} S_{t_n})\frac{T-t}{N},$$

where $t_0 = t, t_1, t_2, \cdots, t_N = T$ are time steps during the period $[t, T]$ and $\triangle_{t_n}$ is the delta value of the option portfolio. In this study, we do not take transactions costs in the delta-neutral hedge operations with spot contracts into consideration because the effects of those transactions costs to the DHGL results are actually negligible due to the high liquidity and the low level of such costs in the USD-JPY exchange rate market. [8].

4 An Empirical Analysis

4.1 Estimation Result for the Model Parameters

Fig.6, Fig.7 and Fig.8 in Appendix show the time series results of the estimated parameters, $\tilde{\kappa}$, $\tilde{v}$, and $\tilde{\rho}$, respectively, in (11), which represents the Heston[1993] stochastic volatility model. These parameters are estimated with the maximum likelihood methodology proposed by Aït-Sahalia and Kimmel[2007]. In particular, Fig.8 shows the time series of estimated $\tilde{\rho}$, which is the correlation between volatility changes and changes in the exchange rate, and we find that the level of $\tilde{\rho}$ is almost negative during the period under consideration.

Fig.1 shows the time series of the volatility risk premium parameter λ estimated with (15). The λ is also almost negative during the period under consideration and the result of negative market volatility risk premium is consistent with the result provided by Low and Zhang[2005]. However, this time series of the λ does not have a time consistency and it moves to negative values significantly after the Sub-prime crisis in 2007 followed by the largest negative period during the Lehman-crisis between September 2008 and October 2008.

[8] As we mentioned before, we take account of transactions costs only in selling the option contracts in our empirical study.

We also show the statistical significance on the level of the volatility risk premium parameter λ exhibited in Fig.1. Table 1 summarizes the statistics for the λ. The top row of Table 1 shows the statistics on the λ for overall period between October 31, 2003 to May 31, 2010, and the middle and the bottom rows show the same statistics for the first half period between October 31, 2003 and December 29, 2006 and the following half period between January 2, 2007 and May 31, 2010, respectively. The second column of Table 1 shows the number of observations in each period. In the third column, we show the percentage of the λ values that are negative, and in the period from January 2, 2007 to May 31, 2010, we can find that almost all the λ have negative values. The percentage of negative values is 94.2 % in that period. The unconditional means and the standard deviations of the λ are listed in the fourth and the fifth columns of Table 1 respectively.

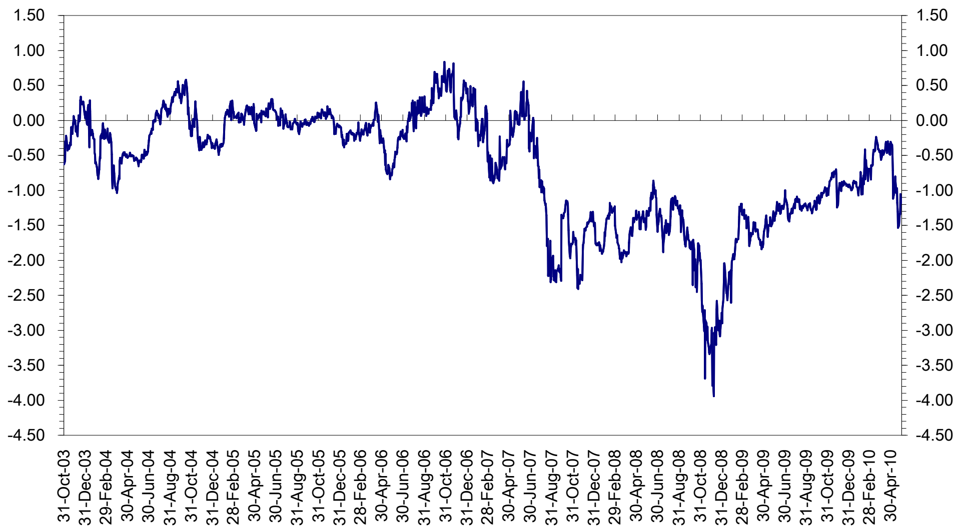

Fig. 1: The time series of the volatility risk premium parameter λ

This figure shows a time series result of the λ estimated at each time point based on the equation (15) from October 31, 2003 to May 31, 2010. The model parameters in the equation (11) are estimated by the maximum liklihood method proposed by Aït-Sahalia and Kimmel[2007], and we update the model parameters daily based on historical 1,750 days daily data with a rolling estimation procedure. We calculate the integral term in the equation (15) by a discretizaion and the numerical integral technique. We interpolate implied volatilities at different moneyness levels with a polynomial approximation methodology proposed by Brunner and hafner[2003] to obtain a fine curve of implied volatilities.

The sample mean is negative for each period, while the high standard deviations of the estimated λ make the means appear to not be significantly different from zero. However, it is misleading to use the unconditional standard deviation to test the mean because serial correlation in the time series of λ can cause the standard deviation to be a biased measure of the actual random error. The next three columns in Table 1 show that the first three autocorrelation coefficients are quite large and decay slowly. This result indicates that the time series may follow an autoregressive process. We also show the partial autocorrelation coefficients in Table 1. The first-order partial autocorrelation coefficient is large in all cases, while the second- and third-order autocorrelation coefficients become much smaller. The pattern for both autocorrelation coefficients and partial autocorrelation coefficients suggests the fitting of an autoregressive process of order three (AR(3)) to the time series of the λ. The AR(3) process for the volatility risk premium parameter can be represented by the following model

$$\lambda_t = \alpha + \beta_1 \lambda_{t-1} + \beta_2 \lambda_{t-2} + \beta_3 \lambda_{t-3} + \epsilon_t,$$

where λ_t is the time-t volatility risk premium parameter and ϵ_t is a white noise process. Its unconditional mean is given by the following formula

$$\mathbb{E}[\lambda_t] = \frac{\alpha}{1 - \beta_1 - \beta_2 - \beta_3},$$

which implies that the null hypothesis of a zero unconditional mean is equivalent to the null hypothesis that the intercept of the AR(3) process is equal to zero.

Table 1: Summary statistics on the volatility risk premium parameter λ

Period	No. of obs.	% of $\lambda <$ 0	Spl. Mean	Spl. Std. Dev.	(1) Auto Corr.			(2) Partial Auto Corr.			(3) AR3 Int.	
					*Lag*1	*Lag*2	*Lag*3	*Lag*1	*Lag*2	*Lag*3	α	$pVal$
Total	1,717	75.3 %	-0.686	0.831	0.989	0.983	0.976	0.989	0.171	0.037	-0.006	0.046
(A)	826	55.0 %	-0.068	0.324	0.957	0.925	0.897	0.957	0.099	0.053	-0.001	0.362
(B)	891	94.2 %	-1.260	0.742	0.979	0.965	0.951	0.979	0.169	0.033	-0.025	0.004

The sample period is from October 31, 2003 to May 31, 2010. Period (A) in this table is the first half period from October 31, 2003 to December 29, 2006 and Period (B) is the following half period from January 2, 2007 to May 31, 2010. The second column of this table shows the number of observations in each time series. In the third column, we present the percentage of the λ which has a negative value. The unconditional means and standard deviations of the λ are respectively exhibited in the fourth and the fifth columns in this table.

We estimate the parameters of the AR(3) process introduced above and show the estimated intercept and its p-value for the t-statistic in the last two columns of Table 1. The intercept is significantly negative at the 5 % level in overall period under consideration and significantly negative at the 1 % level during the second half period from January 2, 2007 to May 31, 2010, whereas it is insignificant, although negative, during the first half period from October 31, 2003 to December 29, 2006. These results, that is to say that the volatility risk premium parameter λ is negative, are consistent with Low and Zhang[2005] in which provides an evidence of the negative volatility risk premium in currency option markets, but are also clearly show that the volatility risk premium parameter λ in the USD-JPY currency option market does not have a time consistency but essentially has a stochastic nature. This feature of time-varying volatility risk premium parameter (or time-varying volatility risk premium) in currency option markets is not found by Low and Zhang[2005] and we should notice that the existence of volatility risk premium in the USD-JPY currency option market is not necessarily significant at any time.

In equilibrium, we can obtain an explicit relation between the risk aversion parameter for the representative agent and the volatility risk premium parameter. According to Heston[1993],

$$-\lambda\tilde{\sigma}_t = \mathrm{Cov}_t\Big(\frac{d\Lambda_t}{\Lambda_t}, d\tilde{\sigma}_t\Big), \tag{18}$$

where Λ_t is the stochastic discount factor process in the dynamic equilibrium setting and in the case that the utility function of the representative agent assumes the following power utility form,

$$U_t = \exp(-\delta t)\frac{S_t^{1-\gamma}}{1-\gamma},$$

where δ is the subjective discount factor and γ is the risk aversion parameter, we can obtain a representation of $\Lambda_t = \exp(-\delta t)S_t^{-\gamma}$ easily. Thus, by (11) and Ito's lemma, the following equation can be derived:

$$\mathrm{Cov}_t\Big(\frac{d\Lambda_t}{\Lambda_t}, d\tilde{\sigma}_t\Big) = -\gamma\tilde{\rho}\tilde{v}\tilde{\sigma}_t. \tag{19}$$

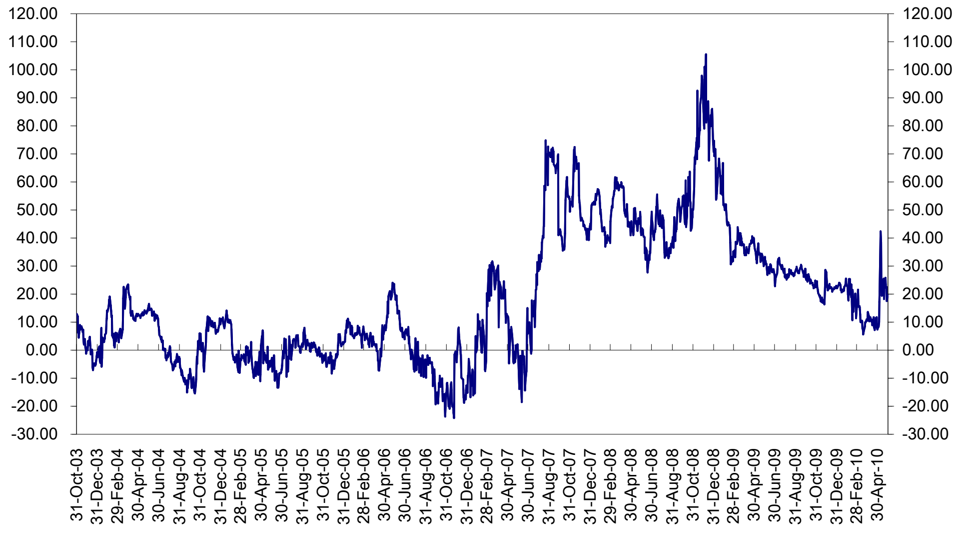

Fig. 2: Risk aversion parameter γ

The power utility is assumed to the utility function for the representative agent, and this figure shows the time series of the risk aversion parameter estimated by the relation of $\gamma = \lambda/(\tilde{\rho}\tilde{v})$ during the period from October 2003 to May 2010. The model parameters in the equation (11) are estimated by the maximum liklihood method proposed by Aït-Sahalia and Kimmel[2007] and we update the model parameters daily based on historical 1,750 days daily data with a rolling estimation procedure. The value of λ is from the result exhibited in Fig.1.

Equations of (18) and (19) lead to an explicit relation between the volatility risk premium parameter λ and the risk aversion parameter γ, that is to say, $\gamma = \lambda/(\tilde{\rho}\tilde{v})$.

This equation enables us to employ an empirical analysis to estimate the time series of the γ under the equilibrium mentioned above. Fig.2 shows the estimation result on the time series of the γ. This figure shows that it moves to significantly positive values after the sub-prime crisis in 2007 followed by the longest positive period during the Lehman crisis from September 2008 to October 2008. Then, it moves down slowly but seems to rise again during the period of the European financial crisis from April 2010 to May 2010.

4.2 A Contribution Analysis on the Expected DHGL

To demonstrate a contribution analysis on the expected DHGL represented by (10), we first calculate $C_t^M[0] - C_t^M[\lambda]$, which is the second term on the right side of the equation (10), using the set of estimated parameters $(\tilde{k}, \tilde{v}, \tilde{\rho})$ and the volatility risk premium parameter λ shown in Fig.1. Combining this calculation result of $C_t^M[0] - C_t^M[\lambda]$ with the DHGL estimated with simulated delta-hedged option returns according to (10), the effect of parameter uncertainty, the first term on the right side of the equation (10), can be estimated.

Table 2: Contribution Analysis for the Expected Delta-hedged Gain Loss

(1) Interval of λs	(2) No. of Obs.	(3) Mean of λ	(4) Expected DHGL: ATM Straddle Short (4-1) RP	(4-2) PU	(4-3) Total	(5) Expected DHGL: OTM delta-25 Put Short (5-1) RP	(5-2) PU	(5-3) Total
Overall	1,717	-0.69	9.7	-3.8	5.9	3.6	2.2	5.8
$\lambda \in (-4, -3.5]$	4	-3.76	89.3	73.2	162.4	32.8	53.1	86.0
$\lambda \in (-3.5, -3]$	18	-3.15	66.6	37.7	104.3	24.3	63.4	87.6
$\lambda \in (-3, -2.5]$	25	-2.78	59.2	62.9	122.1	21.8	57.1	78.9
$\lambda \in (-2.5, -2]$	62	-2.22	34.5	12.7	47.2	12.7	19.5	32.2
$\lambda \in (-2, -1.5]$	186	-1.73	25.5	8.2	33.7	9.5	-0.6	8.9
$\lambda \in (-1.5, -1]$	304	-1.27	15.2	4.9	20.1	5.7	-3.3	2.4
$\lambda \in (-1, -0.5]$	233	-0.75	7.8	-3.1	4.7	2.9	0.9	3.7
$\lambda \in (-0.5, 0]$	461	-0.03	0.6	-26.2	-25.6	0.2	1.6	1.9
$\lambda \in (0, +0.5]$	388	0.17	-1.3	-5.4	-6.7	-0.4	3.6	3.2
$\lambda \in (+0.5, +1]$	36	0.62	-4.2	-9.9	-14.1	-1.5	-0.5	-2.0

Starting on our simulation at the date of October 31, 2003, we compute the DHGL of each strategy at the end date after one month and repeat the same computations on the following day after October 31, 2003. The final simulation starts at May 31, 2010 and ends at June 30, 2010. We finally collect the DHGL results of 1,717 samples for each delta-hedged option strategy through those iterated simulations. We employ the Garman-Kohlhagen model, as an extension of the Black-Scholes model to currency options, to compute the delta of the short option positions for tractability. In this table, "No. of Obs." means the number of samples in each interval for the λ, and "Mean of λ" shows the average of the λ in each interval. "RP" shows the contribution induced by the volatility risk premium to the total expected DHGL and "PU" shows the contribution induced by parameter uncertainty to the total expected DHGL. These attribution results shown in this table are monthly based basis point returns.

Table 2 reports the contribution of the volatility risk premium ("RP" in Table2), which is represented by the second term on the right side of the equation (10), and the contribution of parameter uncertainty ("PU" in Table2), which is represented by the first term on the right side of the equation (10), to the expected DHGL estimated by our historical simulation. To standardize the influence of the difference of the exchange rate level at each time point, we calculate the return represented as $\Pi^G_{t,t+\tau}/S_t$ for each simulation and recognize this return as a proxy for the DHGL at each time point. To verify the effects of the strike price on the results of the contribution analysis for the expected DHGL, we examine historical simulations of short positions of the OTM delta-25 1M puts as well as the ATM 1M straddles. We segment the results into ten levels in terms of the level of volatility risk premium parameter λ at each time point to ensure the time dependencies of this attribution analysis. In this table, "No. of Obs." means the number of samples and "Mean of λ" shows the average of the λ in each interval. "RP" shows the contribution induced by the volatility risk premium to the total expected DHGL and "PU" shows the contribution induced by parameter uncertainty to the total expected DHGL. These attribution results shown in this table are monthly based basis point returns.

In this table, it is clear that the total level of expected DHGL increases in proportion as the volatility risk premium parameter λ decreases for both the delta-hedged ATM straddle short strategy and the delta-hedged OTM delta-25 put short strategy, and both effects of volatility risk premium and parameter uncertainty on the total expected DHGL make a significant impact on that variation. This result also indicates that, in the case of $\lambda \leq -2.5$, the total expected DHGL is more than 100bp per month for the delta-hedged ATM straddle short strategy and approximately 80bp per month for the delta-hedged OTM delta-25 put short strategy. In particular, the effect of parameter uncertainty on the total expected DHGL seems to be more significant than the effect of volatility risk premium for the OTM delta-25 put strategy. One of the most important implications of this result is that the sign and the level of the expected delta-hedged option returns do not generally explain the existence of the volatility risk premiums. There are additional important factors that make an impact on delta-hedged option returns such as parameter uncertainty, rendering standard hedging-based tests on volatility risk premiums explored by, for example, Bakshi and Kapadia[2003] and Low and Zhang[2005], unreliable. We also find a large negative expected DHGL when the λ is in the interval between -0.5 and 0 for the case of ATM straddle short strategy. This result is generated almost entirely by the effect of parameter uncertainty and is especially affected by several jumps in the USD-JPY exchange rate market during the period of the European financial crisis from April 2010 to May 2010.

To understand the statistical significance of the expected DHGL shown in Table 2, we employ the same methodology as that used for the analysis in Table 1. Table 3 and Table 4 show the statistical significance of the total expected DHGL, the effect of volatility risk premium on the total expected DHGL, and the effect of parameter uncertainty on the total expected DHGL. In particular, Table 3 shows the result for ATM straddle short strategy, and Table 4 shows the result for OTM delta-25 put short strategy. Period (A) in these tables is the first term from October 31, 2003 to December 29, 2006 and Period (B) is the following term from January 2, 2007 to June 30, 2010. The second column lists the number of observations in each time series. In the third column, we report the percentage of the DHGL which has a positive value. The unconditional means of the DHGL are listed in the fourth column. The sample means and intercepts (α) shown in these tables are monthly based basis point returns.

In Table 3, we find that the p-value for the intercept of the AR3-process for the total expected DHGL is 0.222 during the entire period, so that the statistical significance of the total expected DHGL does not seem to be high. However, if we focus on Period (B), the p-value on the intercept of the AR3-process decreases to 0.072 and it is significantly different from zero at the 10 % level. This result seems to be induced by the statistical significance of the volatility risk premium during Period (B) (see the middle panel in Table 3).

In contrast, the statistical significance of the effect of parameter uncertainty on the total expected DHGL does not seem to be high during either period. Although parameter uncertainty might affect the level of the total expected DHGL, the statistical significance does not seem to be high in this empirical analysis.

Table 3: Summary Statistics of the DHGL for the ATM Straddle Short Strategy

Period	No. of Obs.	% of Pos.	Sample Mean	(1) Auto Corr.			(2) Partial Auto Corr.			(3) AR3 Int.	
				$Lag1$	$Lag2$	$Lag3$	$Lag1$	$Lag2$	$Lag3$	α	$pVal$
Panel 1 : DHGLs (RP+PU)											
Total	1,717	55.3 %	5.9	0.875	0.779	0.698	0.875	0.059	0.023	0.7	0.222
(A)	826	47.5 %	-8.3	0.861	0.775	0.704	0.861	0.133	0.044	-1.0	0.862
(B)	891	62.6 %	19.1	0.874	0.772	0.686	0.874	0.036	0.017	2.4	0.072
Panel 2 : The effects of volatility risk premium (RP)											
Total	1,717	75.3 %	9.7	0.986	0.979	0.971	0.986	0.211	0.046	0.1	0.055
(A)	826	55.0 %	0.8	0.951	0.920	0.894	0.951	0.154	0.082	0.0	0.268
(B)	891	94.2 %	17.9	0.979	0.966	0.954	0.979	0.202	0.042	0.3	0.021
Panel 3 : The effects of parameter uncertainty (PU)											
Total	1,717	50.2 %	-3.8	0.863	0.761	0.675	0.863	0.062	0.020	-0.4	0.678
(A)	826	47.0 %	-9.1	0.862	0.779	0.709	0.862	0.137	0.045	-1.1	0.876
(B)	891	53.2 %	1.2	0.863	0.754	0.662	0.863	0.039	0.015	0.3	0.432

The sample period of these simulation results is from October 31, 2003 to June 30, 2010. Period (A) in this table is the first term from October 31, 2003 to December 29, 2006 and Period (B) is the following term from January 2, 2007 to June 30, 2010. The second column of this table lists the number of observations in each time series. In the third column, we report the percentage of the DHGL which has a positive value. The unconditional means of the DHGL are listed in the fourth column. The sample means and intercepts (α) shown in this table are monthly based basis point returns.

Table 4 shows the same result as Table 3 for the delta-hedged OTM delta-25 put short strategy. In this case, the effect of parameter uncertainty on the total expected

DHGL seems to be high relative to the delta-hedged ATM straddle short strategy (see the bottom panel in Table 4). As a result, the statistical significance of the total expected DHGL during the entire period also seems to be high relative to Table 3. These results suggest that the effect of parameter uncertainty on the total expected DHGL for the OTM delta-25 put short strategy is more significant compared with the ATM straddle short strategy.

Table 4: Summary Statistics of the DHGL for the OTM Put Short Strategy

Period	No. of Obs.	% of Pos.	Sample Mean	(1) Auto Corr.			(2) Partial Auto Corr.			(3) AR3 Int.	
				*Lag*1	*Lag*2	*Lag*3	*Lag*1	*Lag*2	*Lag*3	α	$pVal$
Panel 1 : DHGLs (RP+PU)											
Total	1,717	65.8 %	5.8	0.904	0.825	0.757	0.904	0.042	0.024	0.5	0.101
(A)	826	63.3 %	4.2	0.862	0.776	0.707	0.862	0.127	0.053	0.5	0.112
(B)	891	68.0 %	7.3	0.911	0.833	0.765	0.911	0.019	0.019	0.7	0.179
Panel 2 : The effects of volatility risk premium (RP)											
Total	1,717	75.3 %	3.6	0.984	0.976	0.969	0.984	0.235	0.084	0.0	0.052
(A)	826	55.0 %	0.3	0.915	0.874	0.847	0.915	0.221	0.143	0.0	0.254
(B)	891	94.2 %	6.6	0.976	0.964	0.952	0.976	0.219	0.070	0.1	0.020
Panel 3 : The effects of parameter uncertainty (PU)											
Total	1,717	63.2 %	2.2	0.900	0.818	0.746	0.900	0.039	0.022	0.0	0.294
(A)	826	63.2 %	3.9	0.863	0.778	0.709	0.863	0.126	0.054	0.0	0.128
(B)	891	63.2 %	0.7	0.906	0.824	0.752	0.906	0.016	0.017	0.1	0.447

The sample period of these simulation results is from October 31, 2003 to June 30, 2010. Period (A) in this table is the first term from October 31, 2003 to December 29, 2006 and Period (B) is the following term from January 2, 2007 to June 30, 2010. The second column of this table lists the number of observations in each time series. In the third column, we report the percentage of the DHGL which has a positive value. The unconditional means of the DHGL are listed in the fourth column. The sample means and intercepts (α) shown in this table are monthly based basis point returns.

Finally, to investigate the magnitude of the effect of parameter uncertainty on the total expected DHGL in the pre- and post-financial crisis periods, we report a subperiod contribution analysis for the total expected DHGL in Table 5 and Table 6 with the same simulation results exhibited in Table 2. We divide the overall period into two periods of pre- and post-Lehman shock: the former period is from October 2003 to September 2008 and the latter period is from October 2008 to June 2010. In Table 5, "VRP Para." denotes the volatility risk premium parameter in each period. For the ATM straddle and the OTM delta-25 put short strategy, "Pre." shows the option premium for each corresponding strategy, and "RP" and "PU" show the contributions of the volatility risk premium and parameter uncertainty on the total expected DHGL, respectively. "Tot." shows the total expected DHGL. Each value introduced above is shown in terms of the average ("Ave.") in each period and also in terms of the relative percentage to the corresponding option premium ("Rel."). The average value of the option premium and each contribution to the total expected DHGL shown in this table are monthly based basis point returns. We obtain our daily delta-hedged option returns by selling a corresponding option and maintaining a delta-neutral portfolio using the spot contracts of the USD-JPY exchange rate until the option matures. In calculating the delta-hedged return of the one-month option sold on a given trading day, say day 0, we use the information for days 1 to 22, assuming that there are 22 trading days before the option maturity date. Then, for the delta-hedged return of the one-month option sold on day 1, we use the information of day 2, day 3, up to day 23. Consequently, the delta-hedged returns of day 0 and day 1 straddles use information from an overlapping period between days 2 and 22. To address the concern that our evidence on a contribution analysis for delta hedged option returns is driven by the common information in the overlapping periods, we also construct a time-series of non-overlapping delta hedged option returns for each option strategy. Specifically, for each option strategy, we construct a monthly series of delta-hedged returns on the one-month options sold at the first trading day of each month in our sample period. Because the delta-hedged returns of the beginning-of-month option only depend on the information of the trading days in the same month, they are non-overlapping. Table 6 reports the result for the non-overlapping returns of each of the two delta-hedged option strategies.

Table 5: Relative Contribution Comparison of the Expected DHGL between Pre- and Post Lehman Crisis : Based on a time-series of overlapping results

	VRP	(A) ATM Straddle				(B) OTM delta-25 Put			
	Para.	(A1) Pre.	(A2) RP	(A3) PU	(A4) Tot.	(B1) Pre.	(B2) RP	(B3) PU	(B4) Tot.
Panel 1 : Overall Period [From October 2003 to June 2010]									
(1) Ave.	-1.37	256.6	9.7	-3.8	5.9	51.0	3.6	2.2	5.8
(2) Rel.	-	100.0 %	3.8 %	-1.5 %	2.3 %	100.0 %	7.0 %	4.4 %	11.4 %
Panel 2 : Pre-Lehman Crisis [From October 2003 to September 2008]									
(1) Ave.	-0.89	217.9	5.1	-13.2	-8.1	42.8	1.9	-1.0	0.9
(2) Rel.	-	100.0 %	2.4 %	-6.1 %	-3.7 %	100.0 %	4.4 %	-2.3 %	2.0 %
Panel 3 : Post-Lehman Crisis [From October 2008 to June 2010]									
(1) Ave.	-2.80	370.9	23.1	24.2	47.3	75.3	8.7	11.8	20.5
(2) Rel.	-	100.0 %	6.2 %	6.5 %	12.7 %	100.0 %	11.5 %	15.6 %	27.2 %

Starting on our simulation at the date of October 31, 2003, we compute the DHGL of each strategy at the end date after one month, and repeat the same computations on the following day after October 31, 2003. The final simulation starts at May 31, 2010 and ends at June 30, 2010. We finally collect the DHGL results of 1,717 samples through those iterated simulations. We employ the Garman-Kohlhagen model, as an extension of the Black-Scholes model to currency options, to compute the delta of the short option positions for tractability. In this table, "VRP Para." denotes the volatility risk premium parameter in each period. "Pre." shows the option premium for each corresponding strategy. "RP" and "PU" show the contribution of the volatility risk premium and parameter uncertainty to the total expected DHGL, respectively. "Tot." shows the total expected DHGL. Each value introduced above is shown in terms of the average ("Ave.") in each period and also shown in terms of the relative percentage to corresponding option premium ("Rel."). The average value of option premium and each contribution to the total expected DHGL shown in this table are monthly based basis point returns.

We find several important pieces of evidence on delta-hedged option returns in Table 5 and 6. First, in the case of the ATM straddle short strategy, the effect of parameter uncertainty on the total expected DHGL does not seem to be high in the overall period from October 2003 to June 2010 because the relative value of the effect of parameter uncertainty to the corresponding option premium is less than 2 % in terms of the absolute value. It is clear that the total expected DHGL in that period is well explained by the effect of the volatility risk premium. However, we should notice that the effect of parameter uncertainty becomes more significant in the post-Lehman shock period, that is to say, from October 2008 to June 2010, because the relative value of that effect to the corresponding option premium is more than 6 % and the relative contribution of that effect to the total expected DHGL is much larger than that of the effect of the volatility risk premium.

Second, in the case of the OTM delta-25 put short strategy, the effect of parameter uncertainty on the total expected DHGL is more significant in the overall period from October 2003 to June 2010, which differs from the result for the case of the ATM straddle short strategy. In Table 6, we find that the effect of parameter uncertainty is much larger than the effect of the volatility risk premium in the overall period, and in the post-Lehman shock period, that is, from October 2008 to June 2010, the effect of parameter uncertainty on the total expected DHGL in terms of the relative value to the corresponding option premium becomes 13 % or more.

Table 6: Relative Contribution Comparison of the Expected DHGL between Pre- and Post Lehman Crisis : Based on a time-series of non-overlapping results

	VRP	(A) ATM Straddle				(B) OTM delta-25 Put			
	Para.	(A1) Pre.	(A2) RP	(A3) PU	(A4) Tot.	(B1) Pre.	(B2) RP	(B3) PU	(B4) Tot.
Panel 1 : Overall Period [From October 2003 to June 2010]									
(1) Ave.	-1.41	258.7	9.6	0.0	9.6	51.8	3.6	3.7	7.3
(2) Rel.	-	100.0 %	3.7 %	0.0 %	3.7 %	100.0 %	7.0 %	7.2 %	14.1 %
Panel 2 : Pre-Lehman Crisis [From October 2003 to September 2008]									
(1) Ave.	-0.92	218.3	5.1	-12.1	-7.0	43.2	1.9	1.5	3.4
(2) Rel.	-	100.0 %	2.3 %	-5.6 %	-3.2 %	100.0 %	4.3 %	3.5 %	7.8 %
Panel 3 : Post-Lehman Crisis [From October 2008 to June 2010]									
(1) Ave.	-2.76	372.3	22.3	34.1	56.4	75.8	8.5	9.9	18.4
(2) Rel.	-	100.0 %	6.0 %	9.2 %	15.2 %	100.0 %	11.2 %	13.0 %	24.3 %

Starting on our simulation at the date of October 31, 2003, we compute the DHGL of each strategy at the end date after one month, and repeat the same computations on the following day after October 31, 2003. The final simulation starts at May 31, 2010 and ends at June 30, 2010. We finally collect the DHGL results of 1,717 samples through those iterated simulations. We employ the Garman-Kohlhagen model, as an extension of the Black-Scholes model to currency options, to compute the delta of the short option positions for tractability. In this table, "VRP Para." denotes the volatility risk premium parameter in each period. "Pre." shows the option premium for each corresponding strategy. "RP" and "PU" show the contribution of the volatility risk premium and parameter uncertainty to the total expected DHGL, respectively. "Tot." shows the total expected DHGL. Each value introduced above is shown in terms of the average ("Ave.") in each period and also shown in terms of the relative percentage to corresponding option premium ("Rel."). The average value of option premium and each contribution to the total expected DHGL shown in this table are monthly based basis point returns.

From these empirical results, we can recognize that parameter uncertainty makes an significant impact on option premiums, especially in the post-financial crisis period, as well as the volatility risk premium, and the effect of parameter uncertainty seems to induce much higher option premiums. Needless to say, the sign and the level of the expected delta-hedged option returns do not generally explain the existence of volatility risk premiums. It needs to be emphasized that there are additional important factors such as parameter uncertainty that make an impact on delta-hedged option returns, rendering standard hedging-based tests on volatility risk premiums explored by previous works unreliable.

5 Concluding Remarks

In this paper, we provide a novel representation of delta-hedged option returns in a stochastic volatility environment. The representation of delta-hedged option returns in which we propose consists of two terms: volatility risk premium and parameter uncertainty. In an empirical analysis, we examine the delta-hedged option returns based on the historical simulation of a currency option market from October 2003 to June 2010. We find that the delta-hedged option returns for OTM put options are strongly affected by parameter uncertainty as well as the volatility risk premium, especially in the post-Lehman shock period.

This study is the first to provide empirical evidence on the effect of parameter uncertainty on dellta-hedged option returns. However, the analysis examined in this paper constitutes a first step toward a more detailed investigation of the empirical characteristics of delta-hedged option returns. A next step might be a detailed analysis on the impact of jump risk on delta-hedged option returns. A further direction of this study will be to provide a contribution analysis of delta-hedged option returns with the effect of parameter uncertainty, as well as the effects of volatility and jump risk premiums.

Appendix A Proof of Proposition 1.

In order to obtain the DHGL formula for the representative option market maker with the misspecified function G, we replace the Greeks defined for F with those for G in (4) and (5),

$$\Pi^G_{t,t+\tau} = \int_t^{t+\tau} \mathcal{L}G(u, \tilde{S}_u, \tilde{\sigma}_u)du + \int_t^{t+\tau} \eta_u \frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u)dW^2_u. \tag{20}$$

Thanks to (7),

$$\begin{aligned} &\frac{\partial G}{\partial t}(t, \tilde{S}_t, \tilde{\sigma}_t) + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 G}{\partial \tilde{S}^2}(t, \tilde{S}_t, \tilde{\sigma}_t) - r_d G(t, \tilde{S}_t, \tilde{\sigma}_t) + (r_d - r_f)S_t \frac{\partial G}{\partial \tilde{S}}(t, \tilde{S}_t, \tilde{\sigma}_t) \\ &= -(\tilde{\theta}_t - \lambda_t)\frac{\partial G}{\partial \tilde{\sigma}}(t, \tilde{S}_t, \tilde{\sigma}_t) - \frac{1}{2}\tilde{\eta}_u^2 \frac{\partial^2 G}{\partial \tilde{\sigma}^2}(t, \tilde{S}_t, \tilde{\sigma}_t) - \tilde{\rho}_t \tilde{\eta}_t \sigma_t S_t \frac{\partial^2 G}{\partial \tilde{S} \partial \tilde{\sigma}}(t, \tilde{S}_t, \tilde{\sigma}_t). \end{aligned} \tag{21}$$

Substituting (21) into (20), we have

$$
\begin{aligned}
\Pi^G_{t,t+\tau} = & \int_t^{t+\tau} \Big[\frac{1}{2}(\eta_u^2 - \tilde{\eta}_u^2)\frac{\partial^2 G}{\partial \tilde{\sigma}^2}(u, \tilde{S}_u, \tilde{\sigma}_u) \\
& + (\rho_u \eta_u - \tilde{\rho}_u \tilde{\eta}_u)\sigma_u S_u \frac{\partial^2 G}{\partial \tilde{S} \partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u) + (\theta_u - \tilde{\theta_u})\frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u)\Big] du \\
& + \int_t^{t+\tau} \lambda_u \frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u) du + \int_t^{t+\tau} \eta_u \frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u) dW_u^2.
\end{aligned} \tag{22}
$$

Thus, taking expectation to (22) under the physical measure $\mathbb{P}$, the expected DHGL for the representative option market maker can be derived in the following equation:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\big[\Pi^G_{t,t+\tau}\big] = & \int_t^{t+\tau} \mathbb{E}^{\mathbb{P}}\Big[\frac{1}{2}(\eta_u^2 - \tilde{\eta}_u^2)\frac{\partial^2 G}{\partial \tilde{\sigma}^2}(u, \tilde{S}_u, \tilde{\sigma}_u) \\
& + (\rho_u \eta_u - \tilde{\rho}_u \tilde{\eta}_u)\sigma_u S_u \frac{\partial^2 G}{\partial \tilde{S} \partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u) + (\theta_u - \tilde{\theta_u})\frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u)\Big] du \\
& + \int_t^{t+\tau} \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u)\Big] du.
\end{aligned}
$$

□

Appendix B Proof of Proposition 2.

The following equation can be derived by applying Ito's lemma to the market price $C_t^M = G(t, \tilde{S}_t, \tilde{\sigma}_t)$,

$$
\begin{aligned}
C^M_{t+\tau} = C^M_t & + \int_t^{t+\tau} \frac{\partial G}{\partial \tilde{S}}(u, \tilde{S}_u, \tilde{\sigma}_u) d\tilde{S}_u \\
& + \int_t^{t+\tau} \frac{\partial G}{\partial \tilde{\sigma}}(u, \tilde{S}_u, \tilde{\sigma}_u) d\tilde{\sigma}_u + \int_t^{t+\tau} \tilde{\mathcal{D}} G(u, \tilde{S}_u, \tilde{\sigma}_u) du,
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{D}} G(t, \tilde{S}_t, \tilde{\sigma}_t) = & \frac{\partial G}{\partial t}(t, \tilde{S}_t, \tilde{\sigma}_t) + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 G}{\partial \tilde{S}^2}(t, \tilde{S}_t, \tilde{\sigma}_t) \\
& + \frac{1}{2}\tilde{\eta}_t^2 \frac{\partial^2 G}{\partial \tilde{\sigma}^2}(t, \tilde{S}_t, \tilde{\sigma}_t) + \tilde{\rho}_t \tilde{\eta}_t \sigma_t S_t \frac{\partial^2 G}{\partial \tilde{S} \partial \tilde{\sigma}}(t, \tilde{S}_t, \tilde{\sigma}_t),
\end{aligned}
$$

and $C_t^M = G(t, \tilde{S}_t, \tilde{\sigma}_t)$ is also solves the equation (21). Under the two equations of (21) and (23), we can derive the following equation with the market price C_t^M,

$$
\begin{aligned}
C^M_{t+\tau} = C^M_t & + \int_t^{t+\tau} \frac{\partial C_u^M}{\partial \tilde{S}_u} d\tilde{S}_u + \int_t^{t+\tau} \Big(r_d C_u^M - (r_d - r_f) S_u \frac{\partial C_u^M}{\partial \tilde{S}_u}\Big) du \\
& + \int_t^{t+\tau} \lambda_u \frac{\partial C_u^M}{\partial \tilde{\sigma}_u} du + \int_t^{t+\tau} \tilde{\eta}_u \frac{\partial C_u^M}{\partial \tilde{\sigma}_u} dW_u^2.
\end{aligned} \tag{24}
$$

If $C_t^M[\lambda_u]$ denotes the time-t call option price which is consistent with the underlying exchange rate process (6) and the equivalent martingale measure $\mathbb{Q}[\lambda_u]$, the following equation can be derived because of the fact that $C_T^M[\lambda_u] = C_T^M[0]$ at the maturity date.

$$\begin{aligned}
&C_t^M[\lambda_u] + \int_t^{t+\tau} \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{S}_u} d\tilde{S}_u + \int_t^{t+\tau} \Big(r_d C_u^M[\lambda_u] - (r_d - r_f) S_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{S}_u} \Big) du \\
&+ \int_t^{t+\tau} \lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{\sigma}_u} du + \int_t^{t+\tau} \tilde{\eta}_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{\sigma}_u} dW_u^2 \\
&= C_t^M[0] + \int_t^{t+\tau} \frac{\partial C_u^M[0]}{\partial \tilde{S}_u} d\tilde{S}_u + \int_t^{t+\tau} \Big(r_d C_u^M[0] - (r_d - r_f) S_u \frac{\partial C_u^M[0]}{\partial \tilde{S}_u} \Big) du \\
&+ \int_t^{t+\tau} \tilde{\eta}_u \frac{\partial C_u^M[0]}{\partial \tilde{\sigma}_u} dW_u^2.
\end{aligned}$$

Rearranging the above equation with (6) and taking expectation to the rearranged equation, we can obtain the following representation.

$$C_t^M[0] - C_t^M[\lambda_u] = \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{\sigma}_u}\Big] du + r_d \int_t^T \mathbb{E}^{\mathbb{P}}\Big[C_u^M[\lambda_u] - C_u^M[0]\Big] du. \tag{25}$$

First, let us assume $\lambda_u \geq 0$. We have the following inequations on the option premium,

$$C_u^M[\lambda_u] - C_u^M[0] \leq 0 \quad (\forall u \in [t, T])$$

and

$$\frac{\partial}{\partial u} \mathbb{E}^{\mathbb{P}}\Big[C_u^M[\lambda_u] - C_u^M[0]\Big] = \mathbb{E}^{\mathbb{P}}\Big[\frac{\partial}{\partial u}\Big(C_u^M[\lambda_u] - C_u^M[0]\Big)\Big] \geq 0.$$

Thus,

$$\begin{aligned}
C_t^M[0] - C_t^M[\lambda_u] &= \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{\sigma}_u}\Big] du + r_d \int_t^T \mathbb{E}^{\mathbb{P}}\Big[C_u^M[\lambda_u] - C_u^M[0]\Big] du \\
&\geq \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{\sigma}_u}\Big] du + r_d (T-t)\Big(C_t^M[\lambda_u] - C_t^M[0]\Big), \\
\therefore \quad \Big(1 + r_d(T-t)\Big)\Big(C_t^M[0] - C_t^M[\lambda_u]\Big) &\geq \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{\sigma}_u}\Big] du.
\end{aligned}$$

We also have a inequation of $\mathbb{E}^{\mathbb{P}}\Big[C_u^M[\lambda_u] - C_u^M[0]\Big] \leq 0$, so the following inequation can be obtained,

$$C_t^M[0] - C_t^M[\lambda_u] \leq \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{\sigma}_u}\Big] du.$$

Thus we can derive the following inequation with two inequations derived above,

$$C_t^M[0] - C_t^M[\lambda_u] \leq \int_t^T \mathbb{E}^{\mathbb{P}}\Big[\lambda_u \frac{\partial C_u^M[\lambda_u]}{\partial \tilde{\sigma}_u}\Big] du \leq \Big(1 + r_d(T-t)\Big)\Big(C_t^M[0] - C_t^M[\lambda_u]\Big).$$

In the case of $\lambda_u < 0$, we have the following inequations,

$$C_u^M[\lambda_u] - C_u^M[0] \geq 0 \quad (\forall u \in [t, T])$$

and

$$\frac{\partial}{\partial u}\mathbb{E}^{\mathbb{P}}\Big[C_u^M[\lambda_u] - C_u^M[0]\Big] = \mathbb{E}^{\mathbb{P}}\Big[\frac{\partial}{\partial u}\Big(C_u^M[\lambda_u] - C_u^M[0]\Big)\Big] \leq 0.$$

Thus we can derive the inequation asserted in this Proposition with the similar approach to the discussion explored above. □

Appendix C Time series data

Table 7: Summary Statistics for Implied Volatilities

Statistics	D5Put	D10Put	D15Put	D25Put	D35Put	ATM	D35Call	D25Call	D15Call	D10Call	D5Call
Mean	16.2	15.2	14.6	13.4	12.7	12.0	11.5	11.1	11.0	11.2	11.3
St. Dev.	7.7	6.7	6.4	5.5	5.1	4.6	4.3	3.9	3.9	3.9	4.0
Min	7.4	6.4	6.9	6.5	6.0	5.8	5.5	4.6	4.8	3.9	5.7
1'st q.	10.5	10.2	9.8	9.3	9.1	8.8	8.5	8.4	8.3	8.5	8.4
Median	15.0	14.1	13.5	12.4	11.8	11.3	10.9	10.7	10.5	10.6	10.7
3'rd q.	19.0	17.7	16.9	15.6	14.7	14.0	13.5	13.3	13.3	13.3	13.5
Max	59.7	54.0	53.7	48.3	46.0	43.0	40.4	38.1	37.0	36.1	35.9
Skew	1.9	1.8	1.9	1.8	1.9	1.8	1.7	1.7	1.5	1.5	1.4
Kurt	5.1	4.5	5.4	5.7	6.2	6.3	6.2	6.4	5.3	4.8	4.3

The implied volatility data is from the European type put and call OTC currency options with maturities of one month and strike prices of 5 delta, 10 delta, 15 delta, 25 delta, 35 delta and ATM. The statistics listed in this table are estimated in the period from October 2003 to June 2010.

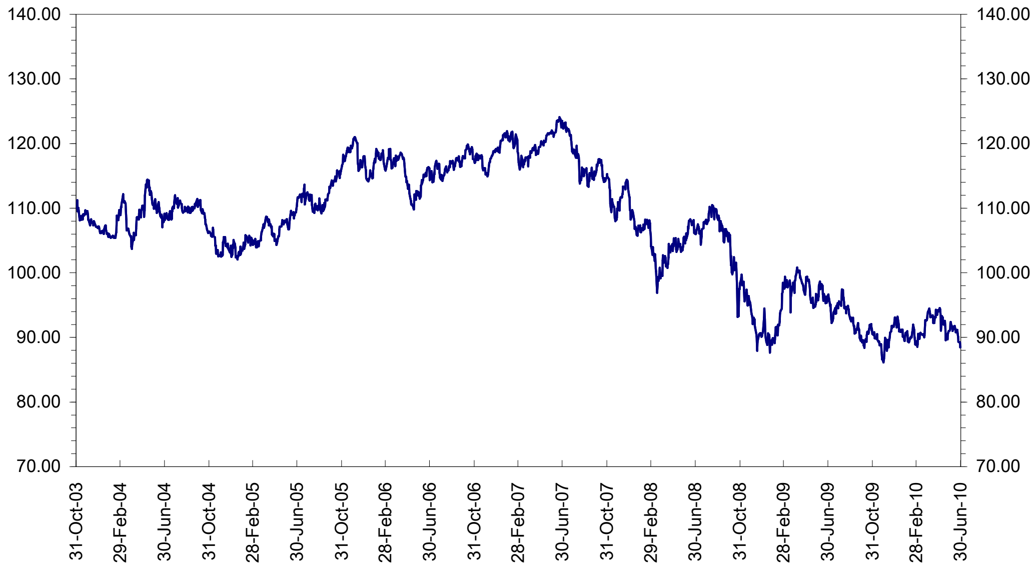

Fig. 3: USD-JPY WM/Reuter Closing Spot Rate

This figure shows the time-series data in the period from October 31, 2003 to June 30, 2010.

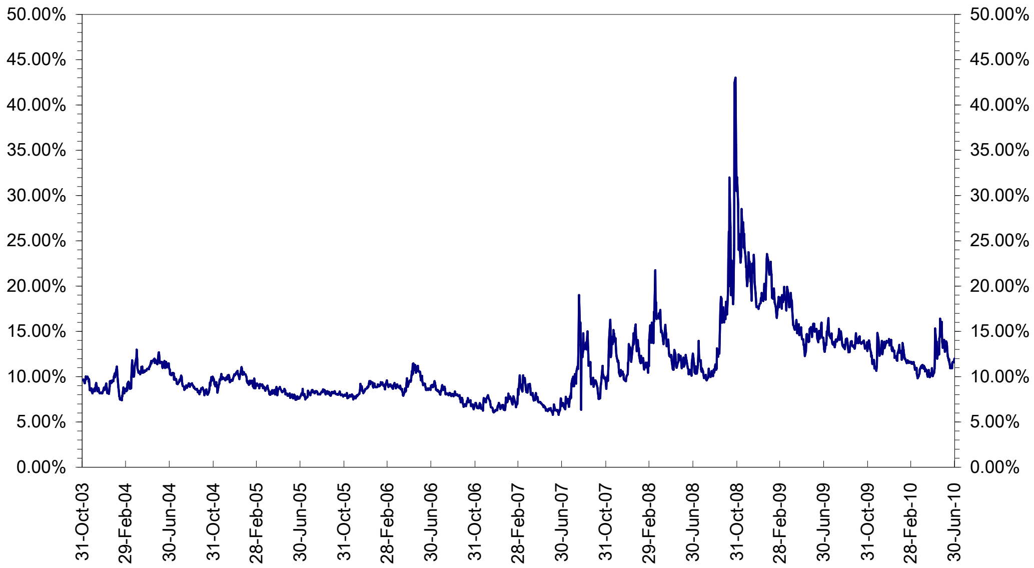

Fig. 4: USD-JPY 1M ATM Implied Volatility (Mid Price)

This figure shows the time-series data in the period from October 31, 2003 to June 30, 2010.

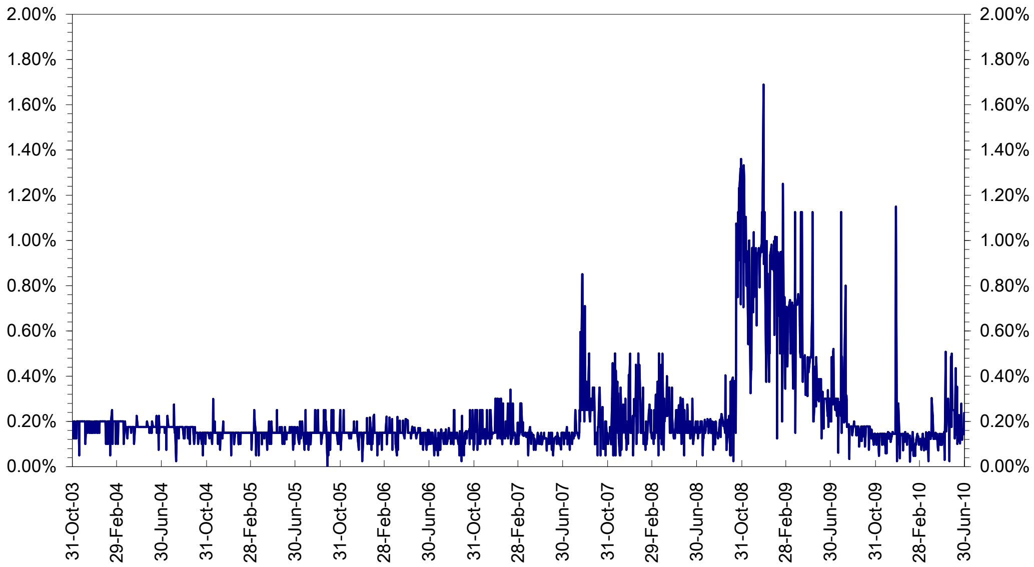

Fig. 5: USD-JPY 1M ATM Mid-Bid Spread

This figure shows the time-series data in the period from October 31, 2003 to June 30, 2010.

Appendix D Parameter estimation results for the Heston[1993] model

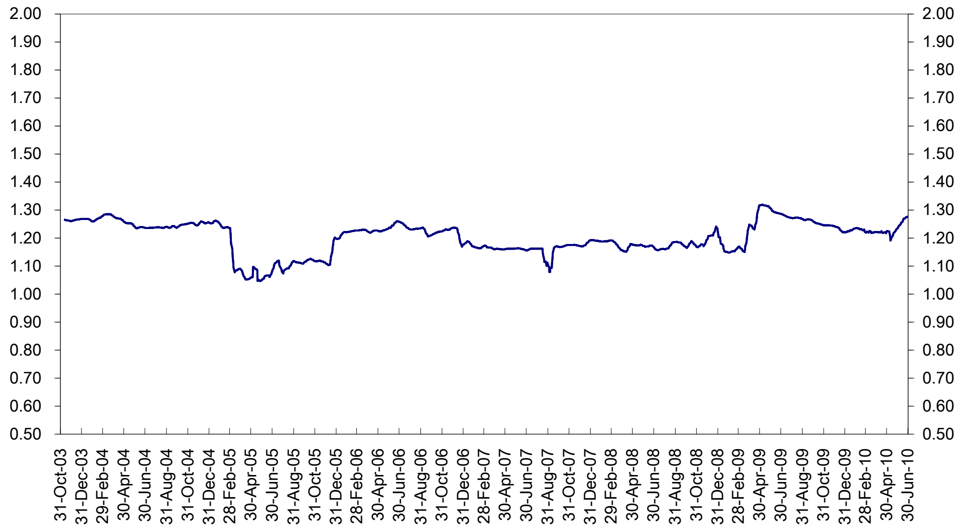

Fig. 6: The time-series of the estimated parameter : $\tilde{k}$

This figure shows estimation results of $\tilde{\kappa}$ at each time point in the period from October 31, 2003 to June 30, 2010. These parameters are estimated with the maximum liklihood method proposed by Aït-Sahalia and Kimmel[2007] and we update the parameters daily based on historical 1,750 days daily data with a rolling estimation procedure.

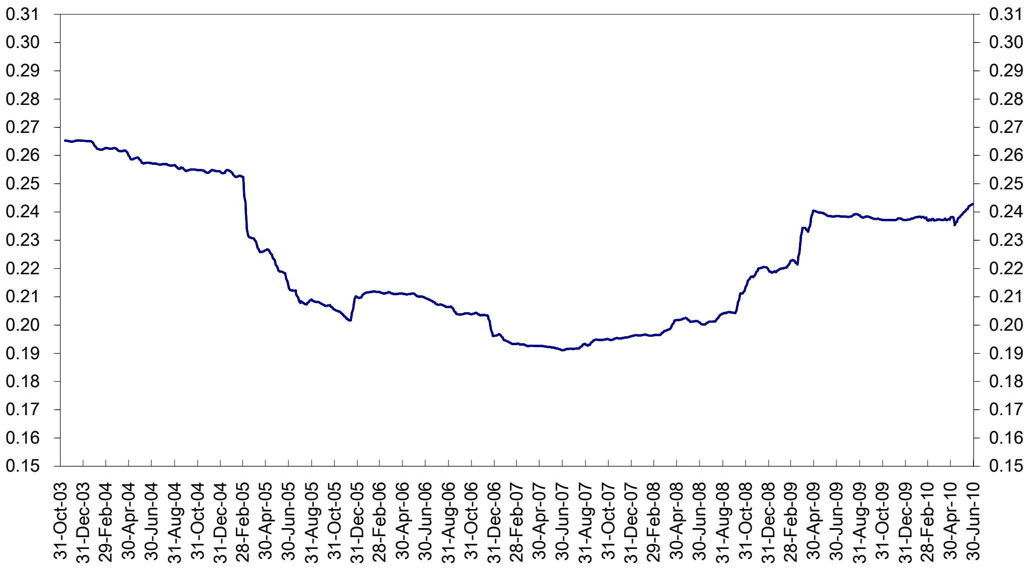

Fig. 7: The time-series of the estimated parameter : $\tilde{v}$

This figure shows estimation results of $\tilde{v}$ at each time point in the period from October 31, 2003 to June 30, 2010. These parameters are estimated with the maximum liklihood method proposed by Aït-Sahalia and Kimmel[2007] and we update the parameters daily based on historical 1,750 days daily data with a rolling estimation procedure.

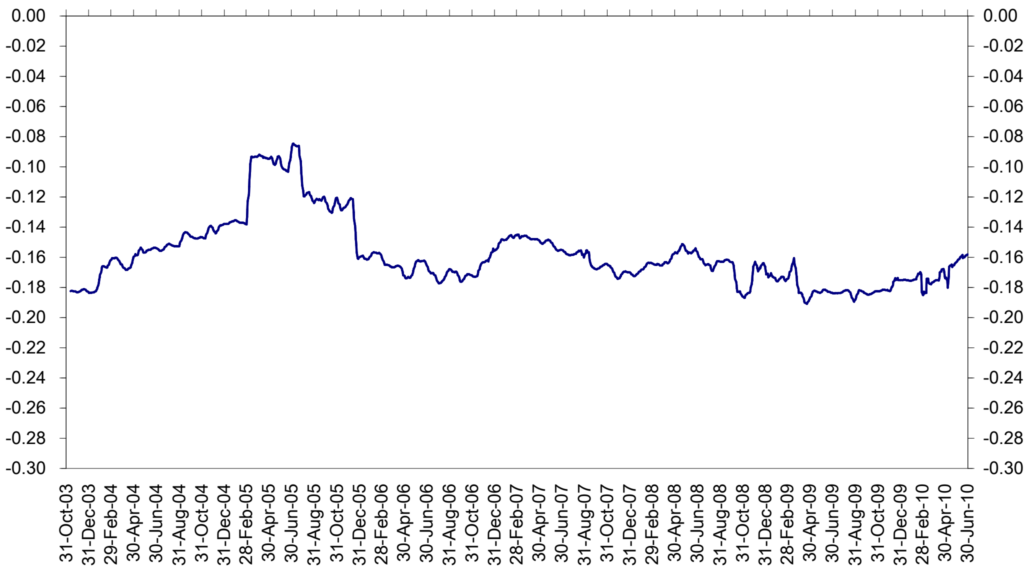

Fig. 8: The time-series of the estimated parameter : $\tilde{\rho}$

This figure shows estimation results of $\tilde{\rho}$ at each time point in the period from October 31, 2003 to June 30, 2010. These parameters are estimated with the maximum liklihood method proposed by Aït-Sahalia and Kimmel[2007] and we update the parameters daily based on historical 1,750 days daily data with a rolling estimation procedure.

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