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Recursive Utility Functions with Extended States

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RECURSIVE UTILITY FUNCTIONS WITH EXTENDED STATES

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ABSTRACT. We introduced a concept called *extended state* that is defined as a history of an observer's recognition on the state of the world Ω with a filtration $\mathbb{G}$. Then Ω is naturally embedded into the set of all extended states $\Omega[\mathbb{G}]$. A subset of $\Omega[\mathbb{G}]$ represents the observer's *ability* to recognize the world.

We applied the concept to dynamic choice theory by calculating value functions that characterize preference relations between consumption plans. The resulting functions are aware not only of prior ambiguity but also of *state ambiguity*.

1. INTRODUCTION

In dynamic choice theory starting from [KP78] through [Str13], we usually think a set of preference relations $\succeq_{t,\omega}$ indexed by a *time* and a *state* $(t, \omega) \in \mathcal{T} \times \Omega$. However, in case Ω is an infinite set, the measure of a singleton set $\{\omega\}$ is (usually) 0. Then, what is the meaning of thinking of a preference relation whose domain has measure 0?

On the other hand, at time t, can an observer pick an *exact* state ω of the world where she lives? Isn't it more natural to assume that she only selects a (possibly non-singleton) *set* of states for representing her belief on the current world?

In order to answer these issues, we replace a single ω by a *narrowing-in* process, called an *extended state*.

An extended state is a history of an observer's recognition on the state of the world where she has lived in. Then, a set of extended states represents her ability to recognize the world. Here, the *ability* is determined not only by her personal talent such as reasoning ability but also determined by her external environment such as constraints delivered by asymmetric information. The important point

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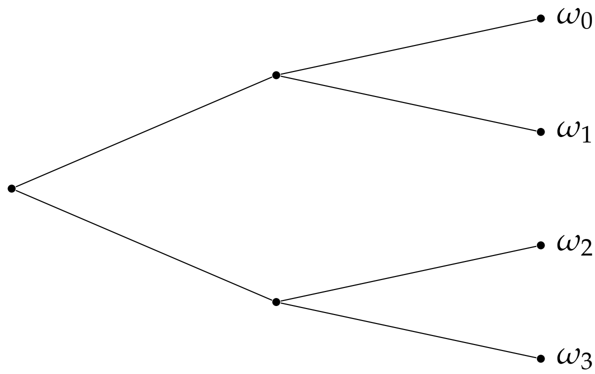

FIGURE 2.1. $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t\in\mathcal{T}})$

is that the ability varies per person. So, we can use the set of extended states to treat some uncertainty or ambiguity just like we do with subjective probabilities. We will see the detail of the concept of extended states in Section 2.

In Section 3, we will apply the concept of extended states to calculate value functions that characterize preference relations between consumption plans. The resulting value functions will be aware not only of usual prior ambiguity but also of state ambiguity. We will also see that the value functions are more conservative than those defined in classical settings.

2. EXTENDED STATES

Let $\mathcal{T}$ be a time domain with the least time 0. For $s, t \in \mathcal{T}$, we write $[s, t] := \{u \in \mathcal{T} \mid s \leq u \leq t\}$. All the discussions in this paper are on a filtered measurable space

$$(\Omega, \mathcal{G}, \mathbb{G} = \{\mathcal{G}_t\}_{t\in\mathcal{T}}) \tag{2.1}$$

that satisfies $\mathcal{G} = \bigvee_{t\in\mathcal{T}} \mathcal{G}_t$.

2.1. **Extended States.**

First, let us see the following simple discrete example that we frequently come back on.

Example 2.1. (1) $\Omega = \{\omega_0, \omega_1, \omega_2, \omega_3\}$,
(2) $\mathcal{T} = \{0, 1, 2\}$,
(3) $\mathcal{G}_0 = \{\emptyset, \Omega\}$,
(4) $\mathcal{G}_1 = \{\emptyset, \{\omega_0, \omega_1\}, \{\omega_2, \omega_3\}, \Omega\}$,
(5) $\mathcal{G} = \mathcal{G}_2 = 2^\Omega$.

We can identify this structure with the binary tree in Figure 2.1 where the time increases from left to right and each node in the tree corresponds to a measurable set in $\mathcal{G}_t$.

The structure in Figure 2.1 may come with a probability measure.

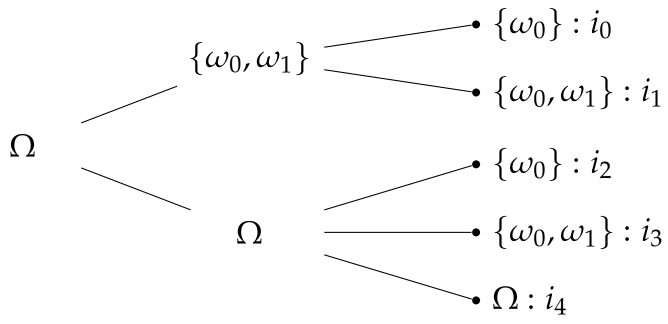

FIGURE 2.2. Narrowing-in processes

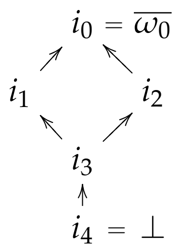

FIGURE 2.3. Partial order among extended states

Suppose that we have two sets A and B in $\mathcal{G}$ with $A \subset B$. Then, the set A represents a finer information or a better recognition about a situation than B does. On the other hand, the fineness of our knowledge at time t about the situation is limited to the sets ranging in $\mathcal{G}_t$.

Now, thinking about a process representing an improvement of our knowledge as time goes by, it is natural to define the process as a decreasing sequence of sets whose member at time t is in $\mathcal{G}_t$. In Figure 2.2, paths i_0 through i_4 are those *narrowing-in* processes called *extended states* when the true state is ω_0.

There are five extended states that share ω_0 as a possible true state. The top of them, i_0 in Figure 2.2, is the most efficient narrowing-in process. On the other hand, In the bottom of them, i_4, is the worst process with no update all the time.

Naturally, we are able to introduce a partial order among these extended states, according to the 'superior-to' relation between them. Figure 2.3 shows the partial order among extended states specified in Figure 2.2.

Here is a formal definition of extended states for general $(\Omega, \mathcal{G}, \mathbb{G})$.

Definition 2.2. [Extended States]

(1) $\Omega[\mathbb{G}] := \{i : \mathcal{T} \to \mathcal{G} \mid (\forall t \in \mathcal{T}) i(t) \in \mathcal{G}_t - \{\emptyset\}$ and $(\forall s, t \in \mathcal{T})[s \leq t \Rightarrow i(s) \supset i(t)]\}$.

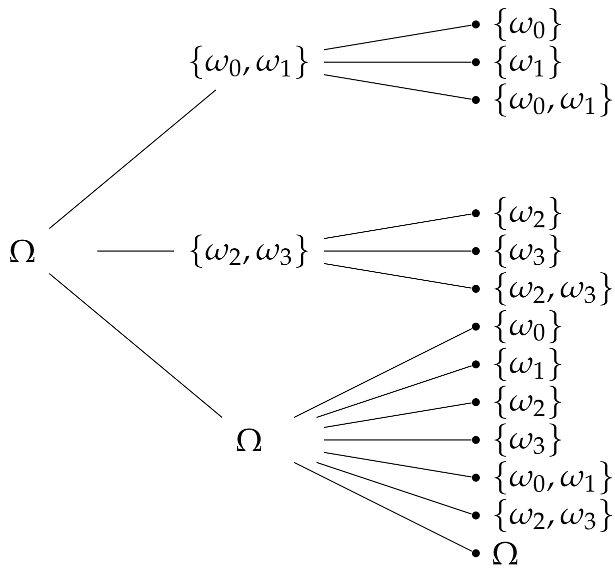

FIGURE 2.4. $\Omega[\mathbb{G}]$

An element of $\Omega[\mathbb{G}]$ is called an *extended state*.

(2) Binary relations $\leq$ and $\leq_t$ on $\Omega[\mathbb{G}]$ are defined by for $i, j \in \Omega[\mathbb{G}]$, $i \leq j \Leftrightarrow (\forall t \in \mathcal{T}) i(t) \supset j(t)$ and $i \leq_t j \Leftrightarrow (\forall s \in [0, t]) i(s) \supset j(s)$.

(3) Let $\perp : \mathcal{T} \to \mathcal{G}$ be a function satisfying $(\forall t \in \mathcal{T}) \perp(t) = \Omega$.
Then $\perp$ is the least element of the partially ordered set (poset) $(\Omega[\mathbb{G}], \leq)$.

Figure 2.4 shows the structure $\Omega[\mathbb{G}]$ corresponding to $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \in \mathcal{T}})$ defined in Figure 2.1. Note that the structure is *not* expected to come with probability measures.

Definition 2.3. (1) For $\omega \in \Omega$ and $t \in \mathcal{T}$, a subset $\overline{\omega}(t) \subset \Omega$ is defined by

$$\overline{\omega}(t) := \bigcap \{A \in \mathcal{G}_t \mid \omega \in A\}. \tag{2.2}$$

(2) For $i \in \Omega[\mathbb{G}]$ and $\omega \in \Omega$, we write $\omega \in i$ if $(\forall t \in \mathcal{T}) \omega \in i(t)$.

Definition 2.4. A filtered measurable space $(\Omega, \mathcal{G}, \mathbb{G} = \{\mathcal{G}_t\}_{t \in \mathcal{T}})$ is called *regular* if $\overline{\omega}(t)$ is in $\mathcal{G}_t$ for any pair of $\omega \in \Omega$ and $t \in \mathcal{T}$.

Note that if Ω is a finite set, $(\Omega, \mathcal{G}, \mathbb{G})$ is always regular for any filtration $\mathbb{G}$.

Remark 2.5. [Embedding of Ω into $\Omega[\mathbb{G}]$] If a filtered measurable space $(\Omega, \mathcal{G}, \mathbb{G})$ is regular, we have

$$\overline{\omega} = \bigvee \{i \in \Omega[\mathbb{G}] \mid \omega \in i\}. \tag{2.3}$$

Moreover, if $\{\omega\} \in \mathcal{G}$ for every $\omega \in \Omega$, then the mapping

$$\begin{array}{ccc} \Omega & \xrightarrow{\bar{\cdot}} & \Omega[\mathbb{G}] \\ \Cup & & \Cup \\ \omega & \longmapsto & \overline{\omega} \end{array}$$

is an embedding. This is why we call an element of $\Omega[\mathbb{G}]$ an *extended state*.

2.2. **Sets of Extended States.** Each observer has a subset of $\Omega[\mathbb{G}]$ corresponding to her narrowing-in ability. Some ability may be determined by external reasons such as the size of the accessible information that came from the asymmetricity of information, whereas some may be determined by internal reasons such as her reasoning power.

Definition 2.6. [Neighborhoods and Regular Subsets]
Let $\mathcal{S}$ be a subset of $\Omega[\mathbb{G}]$.

(1) A *neighborhood* $\mathcal{N}_{\mathcal{S}}(\omega)$ of $\omega \in \Omega$ is a subset of $\mathcal{S}$ defined by

$$\mathcal{N}_{\mathcal{S}}(\omega) := \{i \in \mathcal{S}] \mid \omega \in i\}. \tag{2.4}$$

(2) A subset $\mathcal{S}$ is called *regular* if for every $\omega \in \Omega$, $\mathcal{N}_{\mathcal{S}}(\omega)$ is non-empty, and has a sup in $\mathcal{S}$. We write the sup by $\omega_{\mathcal{S}}$.

Note that $\Omega[\mathbb{G}]$ itself is a regular subset of $\Omega[\mathbb{G}]$ if $(\Omega, \mathcal{G}, \mathbb{G})$ is regular.

Remark 2.7. If a subset $\mathcal{S} \in \Omega[\mathbb{G}]$ is regular and $\{\omega\} \in \mathcal{G}$ for every $\omega \in \Omega$, then the mapping

$$\begin{array}{ccc} \Omega & \longrightarrow & \mathcal{S} \\ \Cup & & \Cup \\ \omega & \longmapsto & \omega_{\mathcal{S}} \end{array}$$

is an embedding.

Definition 2.8 defines a type of the subsets that may not contain some dumb extended states in the original set.

Definition 2.8. [Dominant Subsets] Let $\mathcal{S}$ be a subset of $\Omega[\mathbb{G}]$.

(1) $\Omega_{\mathcal{S}} := \{\omega \in \Omega \mid (\exists i \in \mathcal{S})\omega \in i\}$.
(2) A set $\mathcal{D} \subset \mathcal{S}$ is called a *dominant* subset of $\mathcal{S}$ if it satisfies the following two conditions:
   (a) $\Omega_{\mathcal{D}} = \Omega_{\mathcal{S}}$,
   (b) $(\forall i \in \mathcal{D})(\forall j \in \mathcal{S}) i \leq j$ implies $j \in \mathcal{D}$.

If $\mathcal{D}$ is a dominant subset of $\mathcal{S}$, an observer who has an ability represented by $\mathcal{D}$ is considered to be relatively smarter than an observer who has an ability represented by $\mathcal{S}$.

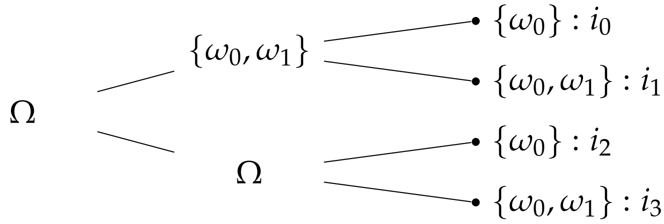

FIGURE 2.5. $\mathcal{N}^2_{\mathbb{G}}(\omega_0)$

Example 2.9. For $\varepsilon > 0$, define $\Omega^\varepsilon[\mathbb{G}]$ by

$$\begin{aligned} \mathcal{N}^\varepsilon_{\mathbb{G}}(\omega) :=& \{ i \in \mathcal{N}_{\Omega[\mathbb{G}]}(\omega) \mid (\forall t \in [\varepsilon, \infty[)\, [i(t) \notin \mathcal{G}_{t-\varepsilon} \\ & \text{or } (\forall A \in \mathcal{G}_t)[\omega \in A \subset i(t) \text{ implies } A = i(t)]]\}, \end{aligned} \tag{2.5}$$

$$\Omega^\varepsilon[\mathbb{G}] := \bigcup_{\omega \in \Omega} \mathcal{N}^\varepsilon_{\mathbb{G}}(\omega). \tag{2.6}$$

Then, $\Omega^\varepsilon[\mathbb{G}]$ is a dominant subset of $\Omega[\mathbb{G}]$.

The tree shown in Figure 2.2 is considered as $\mathcal{N}_{\Omega[\mathbb{G}]}(\omega_0)$. Then, $\mathcal{N}^2_{\mathbb{G}}(\omega_0)$ is a tree in Figure 2.5, which is created just by removing the worst narrowing-in process i_4 from the tree in Figure 2.2.

Next we will think about a situation where an observer has a difficulty to access full information. In other words, the information she can access is limited to a subfiltration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathcal{T}}$ of $\mathbb{G}$, where $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in \mathcal{T}$. Then, it is easy to check that $\Omega[\mathbb{F}] \subset \Omega[\mathbb{G}]$.

One of the examples of the situation comes with a filtration specifying information delay.

Example 2.10. Let $f = \{f_t\}_{t \in \mathcal{T}}$ be a $\mathbb{G}$-*follower process* defined in [AMN13] that is an $\mathcal{T}$-valued $\mathbb{G}$-adapted stochastic process satisfying $f_t \leq t$ and $f_s \leq f_t$ for all pairs s and t in $\mathcal{T}$ with $s \leq t$. Let $\mathbb{G}^f = \{\mathcal{G}^f_t\}_{t \in \mathcal{T}}$ be a *follower filtration* modulated by f introduced in Definition 3.1 of [AMN13] where they define it by

$$\mathcal{G}^f_t := \bigvee_{s \in [0,t]} \mathcal{G}_{f_s}. \tag{2.7}$$

We can see the follower process f as a constraint enforced to the observer. Then the subset $\Omega[\mathbb{G}^f] \subset \Omega[\mathbb{G}]$ is a set of extended states representing her ability.

By combining the results of Example 2.9 and Example 2.10, one of the specifications of subsets of $\Omega[\mathbb{G}]$ is of the form

$$\mathcal{S} := \Omega^\varepsilon[\mathbb{G}^f] \tag{2.8}$$

with a $\mathbb{G}$-follower process f and $\varepsilon > 0$ that can represent both the constraints coming from asymmetric information and the observer's reasoning ability.

We may utilize this type of specifications when we build a set of scenarios systematically for stress tests required by authorized rules such as Basel III.

2.3. **Worlds.**

Definition 2.11. [Worlds] Let $\mathcal{S}$ be a subset of $\Omega[\mathbb{G}]$.

(1) For $t \in \mathcal{T}$ and $i \in \Omega[\mathbb{G}]$, $i^t := i|_{[0,t]}$.

(2) The set of *worlds* denoted by $\mathbb{W}(\mathcal{S})$ is defined by

$$\mathbb{W}(\mathcal{S}) := \{(t, i^t) \mid t \in \mathcal{T}, i \in \mathcal{S}\}. \tag{2.9}$$

(3) For $w = (t, i^t) \in \mathbb{W}(\mathcal{S})$, $\overline{w} := i(t)$.

(4) A binary relation $\leq$ on $\mathbb{W}(\mathcal{S})$ is defined by for $(s, i^s), (t, j^t) \in \mathbb{W}$, $(s, i^s) \leq (t, j^t)$ iff $s \leq t$ and $i^s = j^s$.

Proposition 2.12. *Let $\mathcal{S}$ be a subset of $\Omega[\mathbb{G}]$.*

(1) $(\mathbb{W}(\mathcal{S}), \leq)$ is a poset.

(2) If $\mathcal{G}_0 = \{\emptyset, \Omega\}$, then $(0, \perp^0) \in \mathbb{W}(\mathcal{S})$ is the least element.

Proof. Straightforward. □

If $\mathcal{S} \subset \Omega[\mathbb{G}]$ is a regular set, we have a natural map

$$\begin{array}{ccc} \mathcal{T} \times \Omega & \longrightarrow & \mathbb{W}(\mathcal{S}) \\ \cup & & \cup \\ (t, \omega) & \longmapsto & (t, (\omega_{\mathcal{S}})^t) \end{array}$$

which plays an important role in Section 3.

3. RECURSIVE UTILITY FUNCTIONS

In this section, we assume the time domain is discrete and has its terminal (horizon) time T, that is, $\mathcal{T} = \{0, 1, 2, \ldots, T\}$.

Let X be a Polish space with the Borel σ-field $\mathcal{B}(X)$.

A *consumption plan* is a bounded $\mathbb{G}$-adapted process $h : \mathcal{T} \times \Omega \to X$. Let $\mathcal{H} := \mathcal{H}[\mathbb{G}]$ be a set of all consumption plans.

3.1. **Recursive Utility Functions in Classical Settings.**

Now we proceed to review recursive utility functions in classical settings.

Definition 3.1. [Recursive Utility Functions in Classical Settings]

$V : (\mathcal{T} \times \Omega) \to (\mathcal{H} \to \mathbb{R})$ is a $\mathbb{G}$-adapted process defined by

$$V(t, \omega)(h) = \begin{cases} u(h(t, \omega)) + \beta J(t, \omega)(V(t+1, -)(h)) & \text{if } t < T, \\ u(h(T, \omega)) & \text{if } t = T, \end{cases} \tag{3.1}$$

where

(1) $u : X \to \mathbb{R}$ is a vNM type utility function,
(2) $\beta \in]0, 1[$,
(3) $J : \mathcal{T} \times \Omega \to ((\Omega \to \mathbb{R}) \to \mathbb{R})$.

We provide two examples of J defined in Definition 3.1. They are for the EU model and the MEU model, respectively

Definition 3.2. [Typical J's]

(1) For a probability measure μ on $(\Omega, \mathcal{G})$, $t \in \mathcal{T}$ and $\omega \in \Omega$, define a probability measure $\mu(t, \omega)$ on $(\Omega, \mathcal{G})$ by for $A \in \mathcal{G}$,

$$\mu(t, \omega)(A) := \mu(A \mid \mathcal{G}_t)(\omega). \tag{3.2}$$

(2) **EU model**

$$J(t, \omega)(\xi) := \int_\Omega \xi d\mu_{t,\omega}(t, \omega), \tag{3.3}$$

where $\mu_{t,\omega}$ is a *prior* defined on $(\Omega, \mathcal{G})$.

(3) **MEU model**

$$J(t, \omega)(\xi) := \inf_{\mu \in \mathcal{P}(t,\omega)} \int_\Omega \xi d\mu(t, \omega), \tag{3.4}$$

having *prior ambiguity*, where $\mathcal{P}(t, \omega)$ is a set of probability measures on Ω.

3.2. **Recursive Utility Functions with State Ambiguity.** We want to replace $\mathcal{T} \times \Omega$ appeared in Definition 3.1 by $\mathbb{W}(\mathcal{S})$ in order to allow the value function V to treat *state ambiguity* as well as prior ambiguity.

In the following discussion, let $\mathcal{S} \subset \Omega[\mathbb{G}]$ be a fixed set of extended states. Before defining sets of priors, we need some auxiliary sets that relate to possible next steps from a given world $w \in \mathbb{W}(\mathcal{S})$.

Definition 3.3. Let $w = (t, i^t) \in \mathbb{W}(\mathcal{S})$ and μ be a probability measure on $(\Omega, \mathcal{G})$. μ is said *conditionable* with w if the conditional probability measure $\mu(- \mid \overline{w})$ is well-defined on $\mathcal{G}_t$ [1]. For $\omega \in \Omega$,

$$\mathbf{N}_0(w, \omega) := \{(t+1, j^{t+1}) \mid j \in \mathcal{N}_\mathcal{S}(\omega), j^t = i^t\}, \tag{3.5}$$

$$\mathbf{N}(w, \mu, \omega) := \{v \in \mathbf{N}_0(w, \omega) \mid \mu \text{ is conditionable with } v\}, \tag{3.6}$$

$$\mathbf{D}(w, \mu) := \{\omega \in \overline{w} \mid \mathbf{N}(w, \mu, \omega) \neq \emptyset\}. \tag{3.7}$$

Definition 3.4. [Priors $\mathcal{P}$]

(1) A set-valued function Δ on $\mathcal{G}$ is defined by for $A \in \mathcal{G}$,

$$\Delta(A) := \{\mu \mid \text{a probability measure on } (\Omega, \mathcal{G}) \text{ with } \mu(A) = 1\}. \tag{3.8}$$

[1] We want to avoid situations like the Borel-Kolmogorov paradox.

(2) The set of *extended worlds* denoted by $\mathbf{EW}(\mathcal{S})$ is defined by

$$\mathbf{EW}(\mathcal{S}) := \bigoplus_{w \in \mathbb{W}(\mathcal{S})} \Delta(\overline{w}). \tag{3.9}$$

(3) A subset $\mathcal{P} \subset \mathbf{EW}(\mathcal{S})$ is called to satisfy the *rectangularity condition* if the following four conditions hold:
   (a) for any $w \in \mathbb{W}(\mathcal{S})$, there exists $\mu \in \Delta(\overline{w})$ such that $(w, \mu) \in \mathcal{P}$,
   (b) if $(v, \mu) \in \mathcal{P}$ and $w \in \mathbb{W}(\mathcal{S})$ with $v \leq w$ and μ is conditionable with w, then $(w, \mu(- \mid \overline{w})) \in \mathcal{P}$,
   (c) let $t \in \mathcal{T}$, I be an index set either $\{0, 1, 2, \ldots, N-1\}$ or $\mathbb{N}$, $i, j_n \in \mathcal{S}$ for all $n \in I$ such that $\{j_n(t)\}_{n \in I}$ are mutually disjoint and $\bigcup_{n \in I} j_n(t) = i(t)$. If $((t, i^t), \mu), ((t, j_n^t), \nu_t) \in \mathcal{P}$ for all $n \in I$, then $((t, i^t), \sum_{n \in I} \mu(j_n(t))\nu_n) \in \mathcal{P}$,
   (d) for every $(w, \mu) \in \mathcal{P}$ with $w = (t, i^t)$, $\mathbf{D}(w, \mu) \in \mathcal{G}_t$.

Proposition 3.5. *[Priors $\overline{\mathcal{P}}$] Suppose that $\mathcal{P} \subset \mathbf{EW}(\mathcal{S})$ satisfies the rectangularity condition. Then, if $i(t) = j(t)$, we have $\mathcal{P}(t, i^t) = \mathcal{P}(t, j^t)$, where $\mathcal{P}(w) := \{\mu \mid (w, \mu) \in \mathcal{P}\}$.*

Proof. Let $N := 1$, $j_0 := j$ and $\nu_0 := \nu$ in the condition (c) of Definition 3.4 (3). Then, since $i(t) = j(t)$, the assumptions of the condition (c) are satisfied. Therefore, $((t, i^t), \mu), ((t, j^t), \nu) \in \mathcal{P}$ imply $((t, i^t), \nu) \in \mathcal{P}$. Then, by the condition (a) of Definition 3.4 (3), $\mathcal{P}(t, j^t) \subset \mathcal{P}(t, i^t)$. Similarly we have $\mathcal{P}(t, i^t) \subset \mathcal{P}(t, j^t)$. □

Definition 3.6. When $\mathcal{P} \subset \mathbf{EW}(\mathcal{S})$ satisfies the rectangularity condition, $\overline{\mathcal{P}} : \bigoplus_{t \in \mathcal{T}} \mathcal{G}_t \to \mathbf{Set}$ is a function defined by $\overline{\mathcal{P}}(t, \overline{w}) := \mathcal{P}(w)$.

Note that Definition 3.6 is well-defined by Proposition 3.5.

Example 3.7. Define $\overline{\mathcal{P}}$ over the structure defined in Example 2.1.
(1) $\Theta := [r_L, r_H]$ with $0 < r_L \leq r_H < 1$.
(2) For $(r_0, r_1, r_2) \in [0, 1]^3$, a probability measure on $(\Omega, \mathcal{G})$ $\mu_{r_0 r_1 r_2}$ is defined by the tree in Figure 3.1.
(3) $\overline{\mathcal{P}}(\cdot, \Omega) := \{\mu_{r_0 r_1 r_2} \mid (r_0, r_1, r_2) \in \Theta^3\}$,
   $\overline{\mathcal{P}}(\cdot, \{\omega_0, \omega_1\}) := \{\mu_{1, r_1, 1} \mid r_1 \in \Theta\}$,
   $\overline{\mathcal{P}}(\cdot, \{\omega_2, \omega_3\}) := \{\mu_{0, 1, r_2} \mid r_2 \in \Theta\}$,
   $\overline{\mathcal{P}}(\cdot, \{\omega_i\}) := \{\delta_{\omega_i}\}$.

Definition 3.8. [State-Ambiguity-Aware Recursive Utility Functions]

(1) $V : \mathbf{EW}(\mathcal{S}) \to (\mathcal{H} \to \mathbb{R})$ is a function defined by for $w = (t, i^t) \in \mathbb{W}(\mathcal{S})$, $(w, \mu) \in \mathbf{EW}(\mathcal{S})$ and $h \in \mathcal{H}$,

$$V(w, \mu)(h) = \begin{cases} I(\overline{w}, \mu)h_t + \beta J(w)(h) & \text{if } t < T, \\ I(\overline{w}, \mu)h_T & \text{if } t = T \end{cases} \tag{3.10}$$

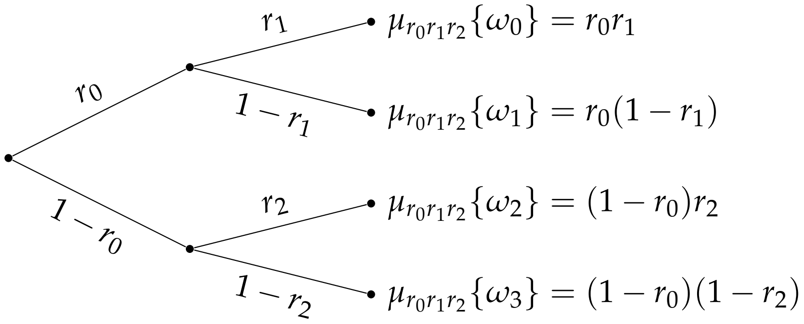

FIGURE 3.1. $\mu_{r_0r_1r_2}$

where
(a) $I : \mathcal{G} \times \Delta(\Omega) \to ((\Omega \to X) \to \mathbb{R})$,
(b) $\beta \in]0, 1[$,
(c) $J : \mathbb{W}(\mathcal{S}) \to (\mathcal{H} \to \mathbb{R})$.

Definition 3.9. [I and J]
The typical I and J for V defined in Definition 3.8 are

$$I(B, \mu)(\xi) := \int_B u \circ \xi d\mu, \tag{3.11}$$

where $u : X \to \mathbb{R}$ is a vNM type utility function, and
(3.12)
$$J(w)(h) := \inf_{\eta \in \mathcal{P}(w)} \int_{\mathbf{D}(w,\eta)} \Big(\inf_{v \in \mathbf{N}(w,\eta,\omega)} V(v, \eta(- \mid \overline{v}))(h) \Big) d\eta(\omega).$$

In general, the values of the new J defined by (3.12) are smaller (or more *conservative*) than those of the old J defined by (3.4) since the new J has an extra inf to pick the minimum value in each neighborhood $\mathbf{N}(w, \eta, \omega)$.

Example 3.10. In this example, we demonstrate how to calculate the value function for a concrete consumption plan $h : \mathcal{T} \times \Omega \to X$ specified on top of the structure in Example 2.1.

Suppose that $u_{t,i} := u(h(t, \omega)) \in \mathbb{R}$ is defined by Figure 3.2 with the dominant set $\mathcal{S} := \Omega^2[\mathbb{G}]$ whose subset $\mathcal{N}^2_{\mathbb{G}}(\omega_0)$ is shown in Figure 2.5.

Here is a backward calculation of $V((1, i^1_0), \mu_{1,r_1,1})(h)$.

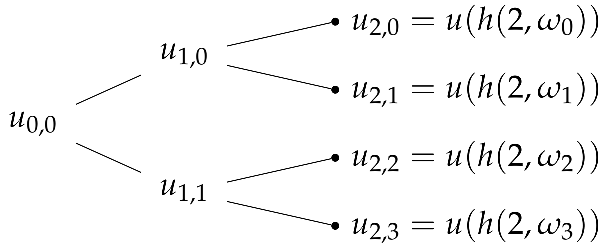

FIGURE 3.2. $u \circ h : \mathcal{T} \times \Omega \to \mathbb{R}$

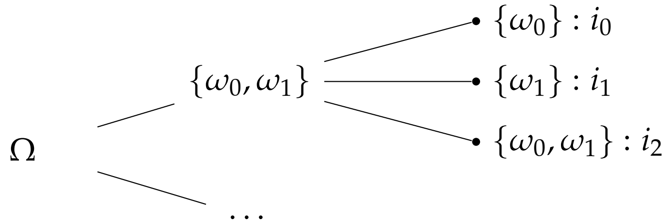

FIGURE 3.3. $V((t, i^t), \mu)(h)$

$$
\begin{aligned}
&V((2, i_0^2), \delta_{\omega_0})(h) = I(i_0(2), \delta_{\omega_0})(h_2) = \int_{\{\omega_0\}} u(h_2(\omega_0))d\delta_{\omega_0} = u_{2,0},\\
&V((2, i_1^2), \delta_{\omega_1})(h) = u_{2,1},\\
&V((2, i_2^2), \mu_{1,r_1,1})(h) = r_1 u_{2,0} + (1 - r_1)u_{2,1},\\
&V((1, i_0^1), \mu_{1,r_1,1})(h) = u_{1,0}\\
&+ \beta \begin{cases} r_L(r_L u_{2,0} + (1 - r_L)u_{2,1}) + (1 - r_L)u_{2,1} & \text{if } u_{2,0} > u_{2,1}\\ r_H u_{2.0} + (1 - r_H)(r_H u_{2,0} + (1 - r_H)u_{2,1}) & \text{otherwise.} \end{cases}
\end{aligned}
$$

3.3. **The Preference for Earlier Resolution of Uncertainty.** In this subsection, we show a trial to investigate what if there is no prior ambiguity but just we have state ambiguity. In our familiar Example 2.1 with the probability measure defined in Example 3.7, this is the case when $r_L = r_H$.

One of the famous puzzles in dynamic choice theory is the *preference for earlier resolution of uncertainty*. Figure 3.4 taken from [Str13], show two consumption plans h and h' that are indifferent under the classical expected utility (EU) value functions. However, once we start thinking the case with prior ambiguity by the MEU model, we have $h \succeq_{0,\omega} h'$.

So, a natural question we have here now is: *can the preference for earlier resolution of uncertainty be represented only by state ambiguity?* In other words, can we make a model representing the following

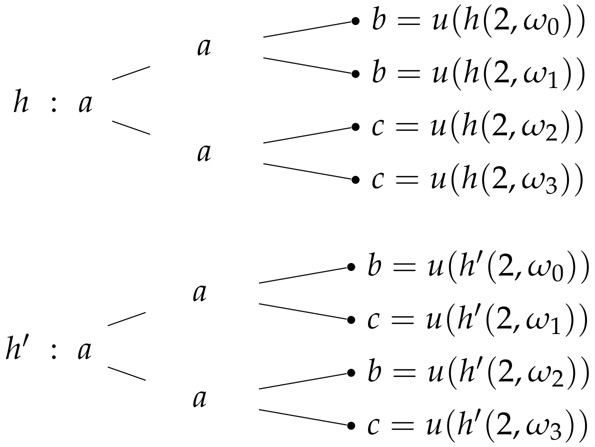

FIGURE 3.4. Consumption plans h and h'

inequation?

$$V((0, \perp^0), \mu)(h) \geq V((0, \perp^0), \mu)(h') \tag{3.13}$$

Figure 3.5 exhibits a calculation of the value function when $r := r_L = r_H$, $\beta = 1$ and $a = 0$ on top of Example 2.1 and Example 3.7, where $rAB := rA + (1 - r)B$ and $A \wedge B := \inf\{A, B\}$ are abbreviations.

As seeing in the figure, we have

$$V((0, \perp^0), \mu)(h) = r(b \wedge rbc)(c \wedge rbc) = V((0, \perp^0), \mu)(h'), \tag{3.14}$$

which is not our expecting result, unfortunately.

4. CONCLUDING REMARKS

We introduced a concept of extended states as a kind of narrowing-in processes of information. We showed that each set of extended states represents an observer's narrowing-in ability determined by her personal talent such as reasoning power and also by constraints coming from her external environment.

As an application of the concept of extended states, we formulated a recursive value function that is aware not only of prior ambiguity but also of state ambiguity. The resulting function is more conservative than a classical value function in the sense that the value of our function is not greater than that of the classical one.

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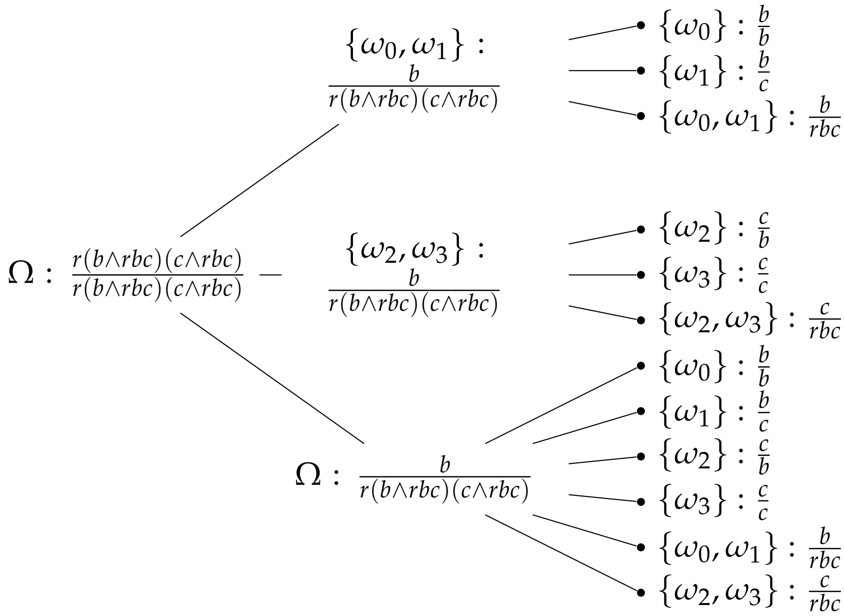

FIGURE 3.5. $\frac{V(h)}{V(h')}$ with $r := r_L = r_H$, $\beta = 1$ and $a = 0$

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