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A method for risk parity/budgeting portfolio based on Gram-Schmidt orthonormalization

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Abstract

We introduce a risk parity/budgeting portfolio using Gram-Schmidt orthonormalization to address problems with two existing risk-based approaches, namely, the asset-based risk parity/budgeting portfolio and the risk budgeting portfolio using principal component analysis. Next, we show and compare the simulation results from the investment strategies based on the new and existing approaches. We observe that the weights of the new portfolio are more interpretable than those of existing ones and that the performance and volatility of the new portfolio are almost the same as those of existing ones, highlighting the advantages of the new approach and overcoming the difficulties of the other approaches.

1 Introduction

For asset allocation, institutional investors use the mean-variance approach developed by Harry Markowitz. This approach is easy to use and simple to explain, even though it is sensitive to the input parameters, particularly expected returns. However, because of high exposure to equities from higher expected returns calculated by long-term historical figures, a number of institutional investors lost substantial amounts of money in the dot-com and 2008 financial crises. These experiences were received as serious warnings that this approach could lead to a lack of diversification and critical damage on their portfolios.

As a solution for this problem, the asset-based Risk Parity Portfolio (RPP) was introduced (Roncali (2013)[3]). This approach does not need any input parameters for expected returns because its underlying idea is to build a portfolio in which risk contribution is the same for all different assets under the investment universe. One criticisms against the RPP is that having identical risk budgets may not mean a diversification of risk sources . An investor may adjust risk budgets to build a portfolio, which is called a Risk Budgeting Portfolio (RBP), but the issue is not directly solvable by risk budgeting.

One way to overcome this challenge is by employing a principal component analysis to extract principal component portfolios by which diversification entropy, the Effective Number of Bets, is calculated (Meucci (2009)[1]). The ENB is maximized through optimization to obtain the most diversified portfolio in terms of entropy. The different principal component portfolios are uncorrelated and can be seen as risk sources. However, this approach may be criticized because the principal component portfolios are not easily interpreted, since they are purely statistical entities as linear combinations of assets weighted by eigenvectors. Also, this approach is inflexible, since only one portfolio is calculated by maximizing the ENB.

Fortunately, principal component portfolios are not the only zero-correlation transformation of original factors that can be used to manage a portfolio.

Meucci et al. (2015) [2] proposes an interpretable definition for de-correlated transformations and its resulting uncorrelated factors. In this study, these researchers select the Minimum-Torsion transformation to minimize tracking errors with respect to the original factors. However, it is unclear whether the small size of the tracking error is sufficient to interpret transformed factors as original factors.

One could think of transforming the original factors in order and not all together. In this paper, we use the Gram-Schmidt orthonormalization to extract understandable risk sources. We also use the same form of objective functions in optimization as the RPP/RBP such that an investor could make flexible investment strategies. This new approach would help investors interpret risk sources to build flexible investment strategies for their various goals.

The remainder of the paper is organized as follows: in Section 2, we briefly revisit the existing frameworks of the RPP/RBP and the Principal Component Risk Budgeting Portfolio (PCRBP). In Section 3, we introduce the Gram-Schmidt Orthonormalization Portfolio (GSOP). In Section 4, we show and compare simulation results of the investment strategies based on these approaches. In Section 5, we present the study's conclusions.

2 Existing Approaches

2.1 Risk Parity/Budgeting Portfolio

In this section, we briefly review how to construct RPP/RBP, following Roncali (2013)[3]. The investment universe consists of n assets. RC_i, $\boldsymbol{w} = (w_1, \ldots, w_n)$ and $\mathcal{R}$ denote the risk contributions of the i^{th} asset, weights and risk measure, respectively. RC_i is defined as follows.

$$RC_i = w_i \frac{\partial \mathcal{R}(\boldsymbol{w})}{\partial w_i}$$

We note by $\boldsymbol{\Sigma}$ the estimated covariance matrix of n assets' returns. When $\mathcal{R}$ is the volatility of a portfolio, $\sigma(\boldsymbol{w}) = \sqrt{\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{w}}$, marginal volatility is

$$\begin{aligned} \frac{\partial \sigma(\boldsymbol{w})}{\partial \boldsymbol{w}} &= \frac{1}{2}(\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{w})^{-1/2}(2\boldsymbol{\Sigma}\boldsymbol{w}) \\ &= \frac{\boldsymbol{\Sigma}\boldsymbol{w}}{\sqrt{\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{w}}} \end{aligned}$$

Therefore, we deduce that

$$RC_i = w_i \cdot \frac{w_i\sigma_i^2 + \sum_{j\neq i} w_j\rho_{i,j}\sigma_i\sigma_j}{\sqrt{\boldsymbol{w}^{\mathrm{T}}\boldsymbol{\Sigma}\boldsymbol{w}}}$$

Roncali (2013)[3] suggests an optimizaton problem below to calculate $\boldsymbol{w}^*$

$$\boldsymbol{w}^* = \arg\min_{\boldsymbol{w}} f(\boldsymbol{w}; \boldsymbol{c})$$

where $\boldsymbol{c}$ denotes risk budgets. Constraints are $\mathbf{1}^{\mathrm{T}}\boldsymbol{w} = 1$ and $0 \leq \boldsymbol{w} \leq 1$. Objective function is defined as follows.

$$f(\boldsymbol{w}; \boldsymbol{c}) = \sum_{i=1}^{n} (w_i \cdot \partial_{w_i}\mathcal{R}(\boldsymbol{w}) - c_i\mathcal{R}(\boldsymbol{w}))^2$$

2.2 Principal Component Risk Budgeting Portfolio

In this section, we review Meucci (2009)[1] and change the form of objective function to that of the RPP/RBP to construct the PCRBP.

We note by $\boldsymbol{\Sigma}$ the estimated covariance matrix of n assets' returns. The principal component analysis derives the following formula.

$$\boldsymbol{E}'\boldsymbol{\Sigma}\boldsymbol{E} = \boldsymbol{\Lambda}$$

where $\boldsymbol{\Lambda} = diag(\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{E} = (e_1, \ldots, e_n)$ represent the vector of eigenvalues of $\boldsymbol{\Sigma}$ and the eigenvectors corresponding to the eigenvalues. $\boldsymbol{E}$

defines n the principal component portfolios, which are uncorrelated, and then variance of the i^{th} principal component portfolio is $\lambda_i (i = 1, 2, \ldots, n)$. Weights of the principal component portfolios are

$$\tilde{\boldsymbol{w}} = \boldsymbol{E}^{-1}\boldsymbol{w}$$

where $\boldsymbol{w} = (w_1, \ldots, w_n)$ denotes the vector of asset weights. When we note by R_p returns of a portfolio,

$$Var(R_p) = \sum_{i=1}^{n} \tilde{w_i}^2 \lambda_i$$

because the different principal component portfolios are uncorrelated. Thus, the proportion of the principal component portfolio's variance to the total variance of the portfolio is

$$\frac{\tilde{w_i}^2 \lambda_i}{Var(R_p)}$$

Meucci (2009)[1] introduces the following entropy of diversification, the ENB, and maximizes it to obtain a portfolio.

$$\begin{aligned} p_i &= \frac{\tilde{w_i}^2 \lambda_i}{Var(R_p)} \\ N_{Ent} &= \exp(-\sum_{i=1}^{n} p_i \ln p_i) \end{aligned}$$

This portfolio construction method is inflexible because only one solution is calculated, but it is the most diversified portfolio in terms of the ENB. In this paper, we adopt the same form of objective functions in optimization as the RPP/RBP to handle investors' various goals.

$$f(\boldsymbol{w}; \boldsymbol{c}) = \sum_{i=1}^{n} (\tilde{w_i}^2 \lambda_i - c_i \cdot Var(R_p))^2$$

3 Risk Parity/Budgeting Portfolio using Gram-Schmidt Orthonormalization

It is difficult to understand the meaning of the principal component portfolios in PCRBP because they are purely statistical entities. To overcome this difficulty, we transform asset returns into uncorrelated ones through Gram-Schmidt orthonormalization and interpret them as their unique movements.

First, we centralize asset returns for covariances of transformed returns to zero. Let the centralized asset returns be $\boldsymbol{a}_i (i = 1, \ldots, n)$. Next, we deduce $\boldsymbol{b}_i$ and $\boldsymbol{u}_i$ using Gram-Schmidt orthonormalization.

$$
\begin{aligned}
\boldsymbol{b}_1 &= \boldsymbol{a}_1 \quad \boldsymbol{u}_1 = \frac{\boldsymbol{b}_1}{|\boldsymbol{b}_1|} = \frac{\boldsymbol{a}_1}{|\boldsymbol{a}_1|} \\
\boldsymbol{b}_2 &= \boldsymbol{a}_2 - (\boldsymbol{a}_2 \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 \quad \boldsymbol{u}_2 = \frac{\boldsymbol{b}_2}{|\boldsymbol{b}_2|} \\
\boldsymbol{b}_3 &= \boldsymbol{a}_3 - (\boldsymbol{a}_3 \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 - (\boldsymbol{a}_3 \cdot \boldsymbol{u}_2)\boldsymbol{u}_2 \quad \boldsymbol{u}_3 = \frac{\boldsymbol{b}_3}{|\boldsymbol{b}_3|} \\
&\cdots \\
\boldsymbol{b}_n &= \boldsymbol{a}_n - (\boldsymbol{a}_n \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 - (\boldsymbol{a}_n \cdot \boldsymbol{u}_2)\boldsymbol{u}_2 - \cdots - (\boldsymbol{a}_n \cdot \boldsymbol{u}_{n-1})\boldsymbol{u}_{n-1} \quad \boldsymbol{u}_n = \frac{\boldsymbol{b}_n}{|\boldsymbol{b}_n|}
\end{aligned}
$$

The order of orthonormalization depends on an investor's viewpoint on markets. If he/she believes that the unique movement of Japanese equity market is captured by the elimination of global equity's effects, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ would be global equity and Japanese equity, respectively.

With a more simple notation, $\boldsymbol{a}_i \cdot \boldsymbol{u}_j = \beta_{i,j} (j = 1, \ldots, n-1)$, we deduce that

$$
\begin{aligned}
\boldsymbol{b}_1 &= \boldsymbol{a}_1 \\
\boldsymbol{b}_2 &= \boldsymbol{a}_2 - \beta_{2,1}\boldsymbol{u}_1 \\
\boldsymbol{b}_3 &= \boldsymbol{a}_3 - \beta_{3,1}\boldsymbol{u}_1 - \beta_{3,2}\boldsymbol{u}_2 \\
&\cdots
\end{aligned}
$$

$$\boldsymbol{b}_n = \boldsymbol{a}_n - \beta_{n,1}\boldsymbol{u}_1 - \beta_{n,2}\boldsymbol{u}_2 - \cdots - \beta_{n,n-1}\boldsymbol{u}_{n-1}$$

Then, R_p is

$$\begin{aligned} R_p &= w_1\boldsymbol{a}_1 + w_2\boldsymbol{a}_2 + \cdots + w_n\boldsymbol{a}_n \\ &= w_1\boldsymbol{b}_1 \\ &+ w_2(\beta_{2,1}\boldsymbol{u}_1 + \boldsymbol{b}_2) \\ &+ w_3(\beta_{3,1}\boldsymbol{u}_1 + \beta_{3,2}\boldsymbol{u}_2 + \boldsymbol{b}_3) \\ &\cdots \\ &+ w_n(\beta_{n,1}\boldsymbol{u}_1 + \beta_{n,2}\boldsymbol{u}_2 + \cdots + \boldsymbol{b}_n) \end{aligned}$$

Because of equation $\boldsymbol{b}_i = |\boldsymbol{b}_i|\boldsymbol{u}_i$,

$$\begin{aligned} R_p &= (w_1|\boldsymbol{b}_1| + w_2\beta_{2,1} + \cdots + w_n\beta_{n,1})\boldsymbol{u}_1 \\ &+ (w_2|\boldsymbol{b}_2| + w_3\beta_{3,2} + \cdots + w_n\beta_{n,2})\boldsymbol{u}_2 \\ &+ (w_3|\boldsymbol{b}_3| + w_4\beta_{4,3} + \cdots + w_n\beta_{n,3})\boldsymbol{u}_3 \\ &\cdots \\ &+ w_n|\boldsymbol{b}_n|\boldsymbol{u}_n \end{aligned}$$

Applying $Var(\boldsymbol{u}_i) = 1$ and $Cov(\boldsymbol{u}_i, \boldsymbol{u}_j) = 0 (i \neq j)$, $Var(R_p)$ is:

$$\begin{aligned} Var(R_p) &= (|\boldsymbol{b}_1|w_1 + \beta_{2,1}w_2 + \cdots + \beta_{n,1}w_n)^2 \\ &+ (|\boldsymbol{b}_2|w_2 + \beta_{3,2}w_3 + \cdots + \beta_{n,2}w_n)^2 \\ &+ (|\boldsymbol{b}_3|w_3 + \beta_{4,3}w_4 + \cdots + \beta_{n,3}w_n)^2 \\ &\cdots \\ &+ (|\boldsymbol{b}_n|w_n)^2 \end{aligned}$$

Therefore, the proportion of an orthonormalized asset's variance to the total variance of the portfolio is

$$\frac{(|\boldsymbol{b}_i|w_i + \beta_{i+1,i}w_{i+1} + \cdots + \beta_{n,i}w_n)^2}{Var(R_p)}$$

We define an objective function as follows:

$$f(\boldsymbol{w};\boldsymbol{c}) = \sum_{i=1}^{n}((|\boldsymbol{b}_i|w_i + \beta_{i+1,i}w_{i+1} + \cdots + \beta_{n,i}w_n)^2 - c_i \cdot Var(R_p))^2$$

4 Empirical Analysis

4.1 Data

In this paper, we use monthly returns from April 2000 to March 2016. Foreign equity (FE), foreign bond (FB), Japanese equity (JE) and Japanese bond (JB) represent MSCI-KOKUSAI (Gross, JPY), Citi WGBI (ex. Japan, JPY), TOPIX (dividend included) and NOMURA-BPI (aggregate), respectively. The data source is Bloomberg. Table 1 shows the descriptive statistical values of these 4 assets.

For further details of these assets' historical behaviors, Figure 1 shows cumulative returns, 36 month rolling volatilities and correlation coefficients for the assets. FE appreciated steadily until 2007 and slipped significantly in 2008 because of the financial crisis. FE moved within the range from 2009 to the middle of 2012 during the European debt crisis followed by a strong recovery caused by global economic expansion. FE's volatility behaved in the reverse direction, that is, its volatility fell when its performance was positive, and vice versa. FB moved in almost the same direction, affected by JPY/USD rate, although its volatility was considerably lower than FE. The correlation coefficient between FB and FE began rising in 2005 and then kept a level above 0.5 since 2008. JE's behavior and volatility was similar to those of FE, while JE lagged behind FE, beginning in 2009. The correlation coefficient between JE and FE/FB surged in 2007, appreciated further in 2008 and kept its level in subsequent years. JB increased in small steps with extremely low volatility. The correlation coefficient between JB and FE started around zero and went into the negative zone in 2007, then kept moving within a range from -0.5 to -0.2. The correlation coefficient between JB and FB was positive until 2007 and then became negative in 2008. The correlation coefficient between JB and JE ranged from -0.5 to 0.0 over the entire period.

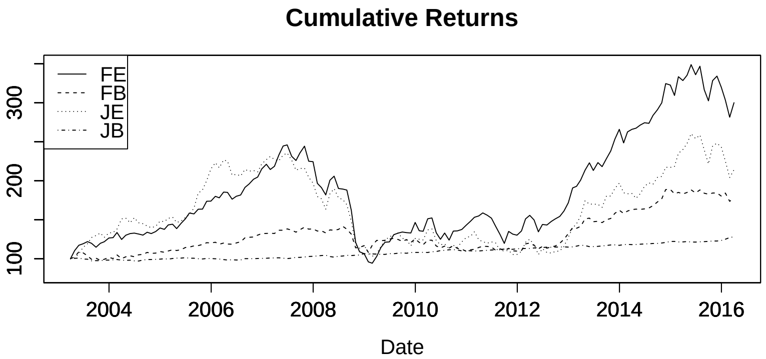
Table 1: Statistics of the 4 Assets

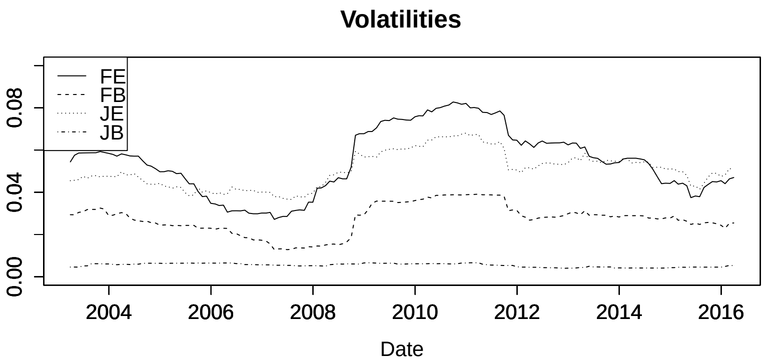
	FE	FB	JE	JB
Return ann. (%)	8.82	4.54	6.03	1.93
Volatility ann. (%)	19.43	9.82	18.27	1.96
Return to Vol. Ratio	0.45	0.46	0.33	0.98
Skewness	-0.87	-0.58	-0.41	-0.38
Kurtosis	2.32	3.24	0.85	1.88
Max	12.87	8.49	12.61	1.72
Min	-25.33	-13.35	-20.26	-2.12
P-value of Shapiro-Wilk Normality Test	0.00011	0.00005	0.09176	0.00140

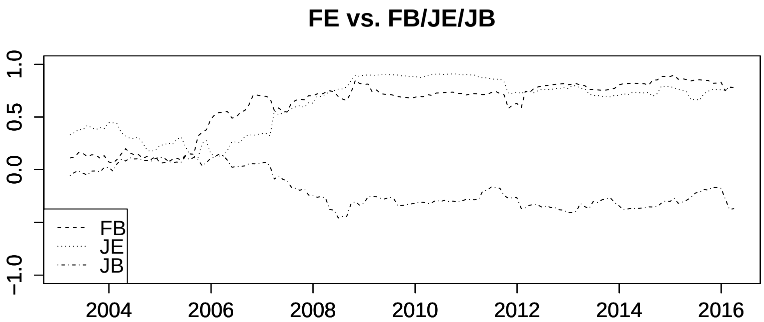
4.2 Methods for Portfolio Construction

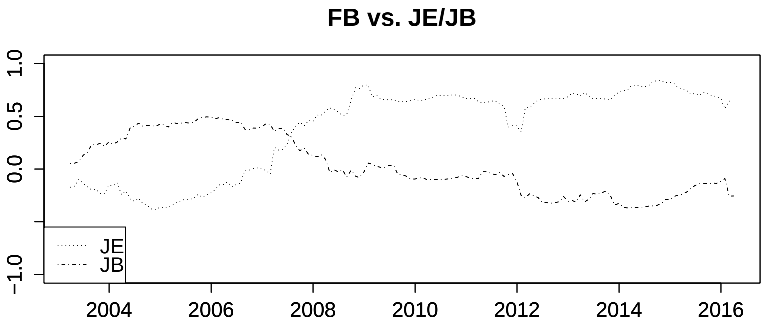
We simulate the Equally Weighted Portfolio (EWP), the RPP/RBP, the PCRBP and the GSOP to compare performance indexes; namely, return, volatility and the return to volatility ratio. We build aggressive and conservative portfolios for the RPP/RBP and the GSOP, applying risk budgets in addition to parity. These portfolios are rebalanced at the end of every month.

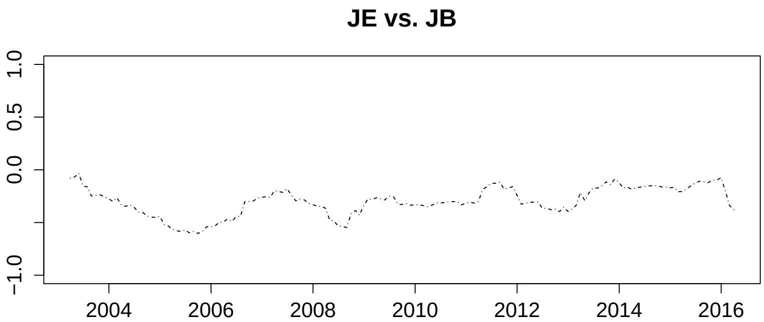
Figure 1: Cumulative Returns, Volatilities and Correlation Coefficients of the 4 Assets






FE vs. FB/JE/JB
1.0
0.5
0.0
−1.0
FB
JE
JB
2004
2006
2008
2010
2012
2014
2016


FB vs. JE/JB
1.0
0.5
0.0
−1.0
JE
JB
2004
2006
2008
2010
2012
2014
2016


JE vs. JB
1.0
0.5
0.0
−1.0
2004
2006
2008
2010
2012
2014
2016

For EWP, the weights of the assets are 25%.

For RPP, we estimate a covariance matrix from the 36 monthly returns at the time of rebalancing. The risk measure is volatility. The constraints are $\mathbf{1}^{\mathrm{T}}\boldsymbol{w} = 1$ and $0 \leq \boldsymbol{w} \leq 1$, and the risk budget is the same for all assets, namely, $c_i = 1/4$. For the RBP Aggressive, the risk budgets of FE, FB, JE and JB are 40%, 30%, 20% and 10%, respectively. For the RBP Conservative, the risk budgets of FE, FB, JE and JB are 10%, 20%, 30% and 40%, respectively. We use the function constrOptim of programming language R[4] for optimization.

For PCRBP, we estimate the covariance matrix from 36 monthly returns at the time of rebalancing. Constraints are the same as the RPP/RBP and the risk budget is the same for all assets, namely, $c_i = 1/4$. We do not simulate portfolios of any other risk budgets because, as mentioned in Section 1, it is difficult to understand the meaning of the principal component portfolios. We use the function prcomp[5] and function constrOptim[4] for the PCA and the optimization, respectively.

For the GSOP, we orthonormalize FE, FB, JE and JB in this order. This approach means that we assume that the unique movement of FE is its own, that the unique movement of FB is obtained by eliminating FE's effect, that the unique movement of JE is obtained by eliminating FE's and FB's effects and that the unique movement of JB is obtained by eliminating the other assets' effects. We estimate the covariance matrix from the 36 monthly returns at the time of rebalancing. The constraints are the same as the RPP/RBP. For the GSOP Parity, the risk budget is the same for all assets, namely, $c_i = 1/4$. For the GSOP Aggressive, the risk budgets of orthonormalized FE, FB, JE and JB are 40%, 30%, 20% and 10%, respectively. For the RBP Conservative, the risk budgets of orthonormalized FE, FB, JE and JB are

10%, 20%, 30% and 40%, respectively. We use the function gramSchmidt[6] and the function constrOptim[4] of R for orthonormalization and optimization, respectively.

- EWP
- RPP
- RBP Aggressive
- RBP Conservative
- PCRBP Parity
- GSOP Parity
- GSOP Aggressive
- GSOP Conservative

4.3 Results

Figure 2 shows the simulated weights of the different portfolios. w_1, w_2, w_3 and w_4 represent the weights of FE, FB, JE and JB, respectively. Figure 3 is used to track the movements of w_1, w_2 and w_3 of the GSOP Parity more clearly.

In Figure 3, w_1 is inversely affected by FE's volatility, which is easy to understand because w_1 is multiplied by $|\boldsymbol{b}_1|$ in the objective function, and any other element is not required to be checked. w_2 has only two coefficients in the objective function, $|\boldsymbol{b}_2|$ and $\beta_{2,1}$, which explains why w_2 has moved in the opposite way of FB's volatility, dropping in 2005 when the correlation coefficient between FB and FE surged. w_3 was inversely affected by

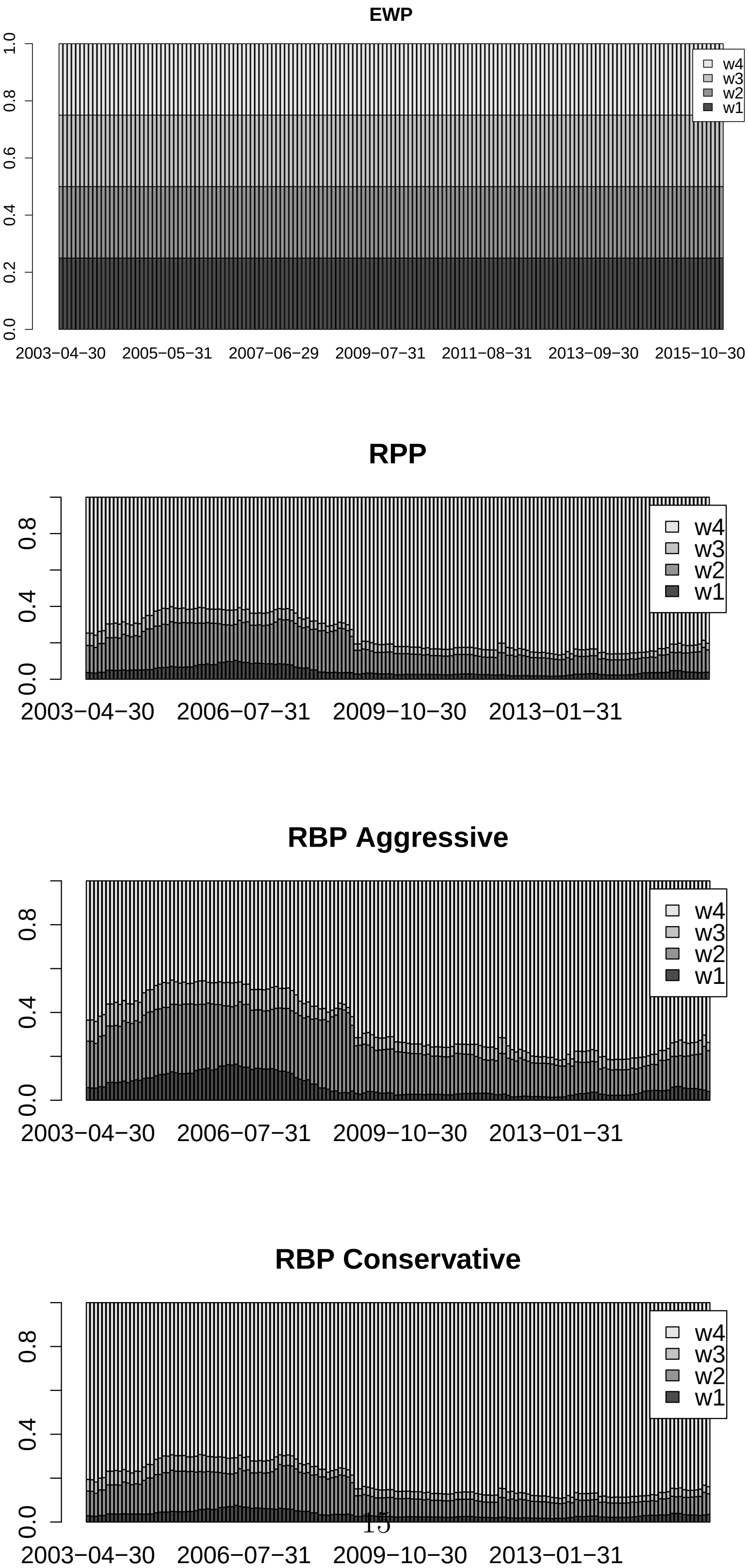
JE's volatility and the correlation coefficient between JE and FE/FB because $|\boldsymbol{b}_3|$, $\beta_{3,1}$ and $\beta_{3,2}$ are coefficients of w_3 in the objective function. JB's volatility was extremely low and generally had negative correlations with the other assets, which made w_4 the highest weight in the portfolio. However, in the RPP/RBP and the PCRBP, w_i has four coefficients in each objective function, which makes it more difficult to understand what the calculated weights of the portfolios mean, in addition to the criticisms mentioned in Section 1.

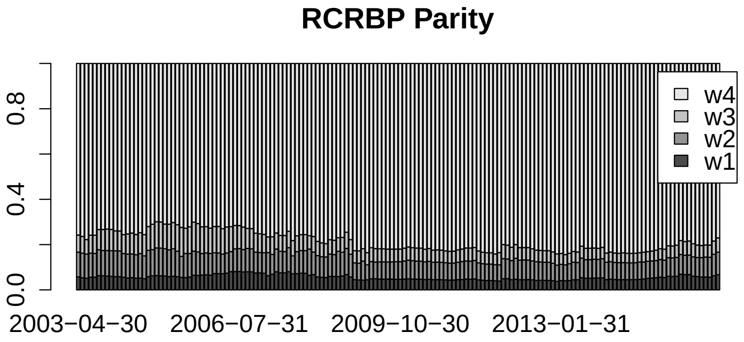
Table 2 and Figure 4 show the performance indexes and cumulative return indexes.

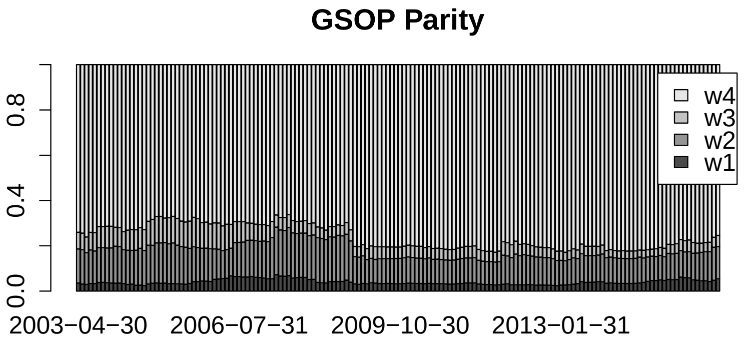
The return and volatility of the GSOP Parity are very close to those of the RPP and the PCRBP Parity. Compared with the GSOP Parity, the GSOP Aggressive has a higher return and volatility, while the GSOP Conservative has a lower return and volatility. In a similar way, compared with the RPP, the RBP Aggressive has a higher return and volatility, while the RBP Conservative has a lower return and volatility. The return and volatility of the GSOP Aggressive are not very different from those of the RBP Aggressive. The return and volatility of the GSOP Conservative are almost identical to those of the RBP Conservative.

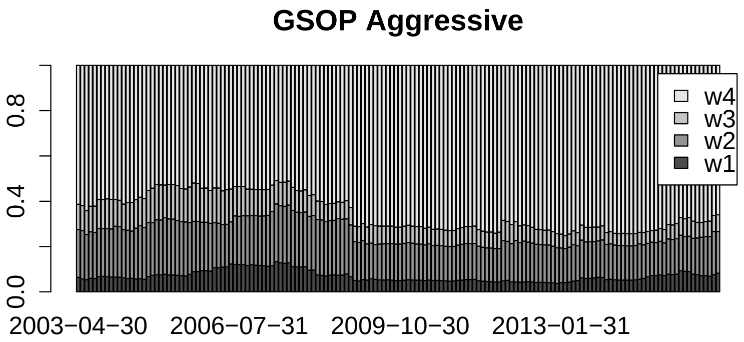
These results suggest that the performance indexes of the GSOP would be close to those of portfolios based on the existing approaches. Additionally, an investor could realize the assumed changes of performance indexes by adjusting their risk budgets.

Figure 2: Weights




RCRBP Parity
0.0
0.4
0.8
w4
w3
w2
w1
2003−04−30
2006−07−31
2009−10−30
2013−01−31


GSOP Parity
0.0
0.4
0.8
w4
w3
w2
w1
2003−04−30
2006−07−31
2009−10−30
2013−01−31


GSOP Aggressive
0.0
0.4
0.8
w4
w3
w2
w1
2003−04−30
2006−07−31
2009−10−30
2013−01−31

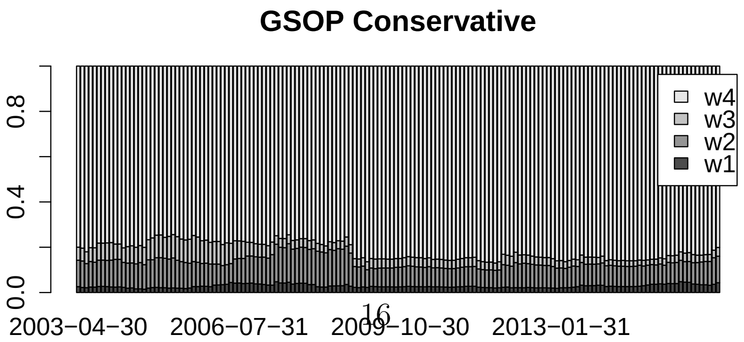
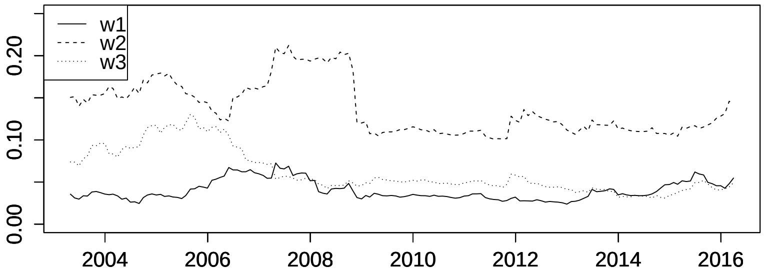

GSOP Conservative
0.0
0.4
0.8
w4
w3
w2
w1
2003−04−30
2006−07−31
2009−10−30
2013−01−31

Figure 3: w_1, w_2 and w_3 of GSOP Parity

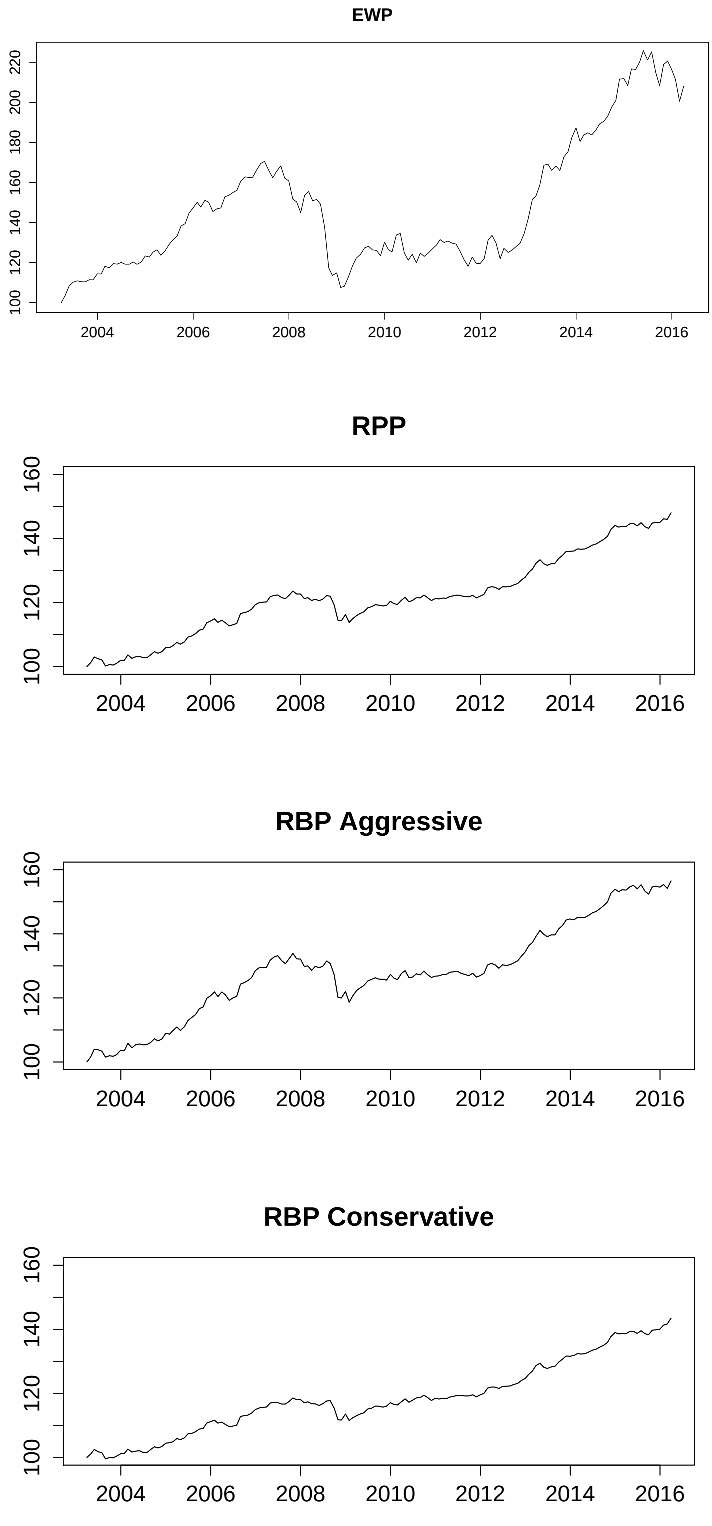


5 Conclusions

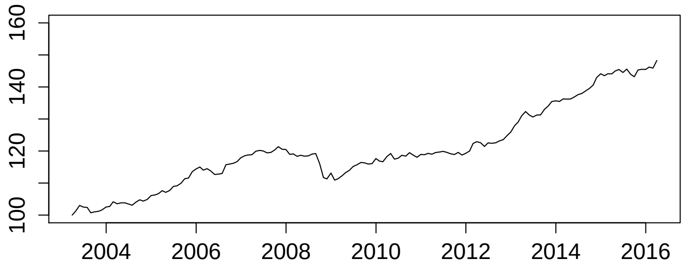
An investor could more easily interpret the calculated weights of the GSOP than the RPP/RBP and the PCRBP. The GSOP's return and volatility would be comparable to those of the RPP/RBP and the PCRBP, and any assumed changes of the performance could be realized by adjusting their risk budgets. These points highlight the GSOP's advantages in that it attempts to diversify risk sources, while the RPP/RBP does not, and that the risk sources are understandable when they are not in the PCRBP.

Further studies should be implemented on the order of orthonormalization because it was not scientifically decided in this paper, assuming that the global market excluding a local market affects the local market and that equity markets affect bond markets. This assumption would be inappropriate, for example, in an event that the bond market triggers turbulence in the equity market for a certain period. Also, the more numbers there are in an investment universe , the more difficult it becomes to determine objectively the order of orthonormalization.

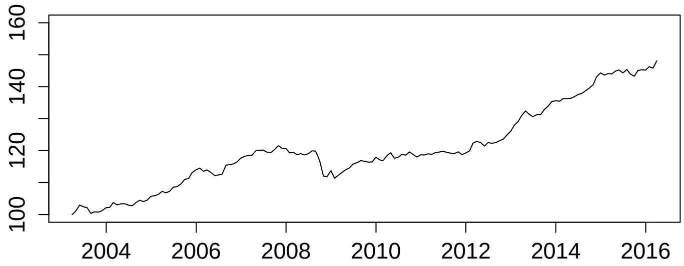
Figure 4: Cumulative Return Indexes



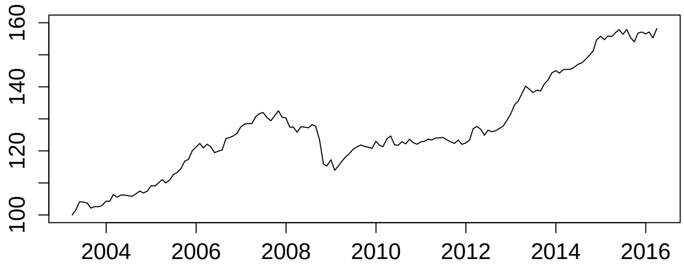
PCRBP Parity



GSOP Parity



GSOP Aggressive



GSOP Conservative

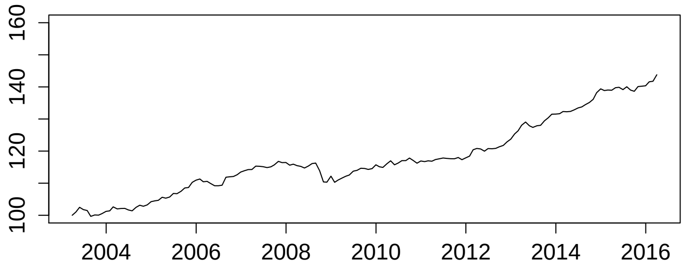


Table 2: Performance Indexes

Indexes	EWP	RPP	RBP Agg.	RBP Con.
Return ann. (%)	5.79	3.06	3.51	2.82
Volatility ann. (%)	10.55	2.82	3.69	2.42
Return to Vol. Ratio	0.55	1.09	0.95	1.16
Skewness	-0.95	-1.09	-1.23	-0.92
Kurtosis	3.26	4.45	5.40	3.89
Max	7.47	2.73	3.16	2.49
Min	-14.63	-4.03	-5.61	-3.15
Indexes	PCRBP Par.	GSOP Par.	GSOP Agg.	GSOP Con.
Return ann.(%)	3.08	3.07	3.59	2.83
Volatility ann.(%)	2.85	2.91	4.08	2.43
Return to Vol. Ratio	1.08	1.05	0.88	1.16
Skewness	-0.95	-1.06	-1.20	-0.83
Kurtosis	3.61	4.44	4.80	3.29
Max	2.44	2.50	2.95	2.27
Min	-3.76	-4.11	-6.04	-3.07

References

[1] Meucci, A. (2009), "Managing Diversification", *Risk*, May 2009

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